Non-trivial, static, geodesically complete space-times with a negative cosmological constant II. $n \geq 5$

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Abstract

We show that the recent work of Lee [23] implies existence of a large class of new singularity-free strictly static Lorentzian vacuum solutions of the Einstein equations with a negative cosmological constant. This holds in all space-time dimensions greater than or equal to four, and leads both to strictly static solutions and to black hole solutions. The construction allows in principle for metrics (whether black hole or not) with Yang-Mills-dilaton fields interacting with gravity through a Kaluza-Klein coupling.

1 Introduction

In recent work [3] we have constructed a large class of non-trivial static, geodesically complete, four-dimensional vacuum space-times with a negative cosmological constant. The object of this paper is to establish existence of higher dimensional analogues of the above.

More precisely, we wish to show that for $\Lambda < 0$ and $n \geq 4$ there exist n–dimensional *strictly static*[1] solutions $(\mathscr{M}, \mathfrak{g})$ of the vacuum Einstein equations with the following properties:

1. $(\mathscr{M}, \mathfrak{g})$ is diffeomorphic to $\mathbb{R} \times \Sigma$, for some $(n-1)$–dimensional spacelike Cauchy surface Σ, with the $\mathbb{R}$ factor corresponding to the action of the isometry group.

2. (Σ, g_Σ), where g_Σ is the metric induced by $\mathfrak{g}$ on Σ, is a complete Riemannian manifold.

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[1] We shall say that a space-time is *strictly static* if it contains a globally timelike hypersurface orthogonal Killing vector field.

3. $(\mathscr{M}, \mathfrak{g})$ is geodesically complete.

4. All polynomial invariants of $\mathfrak{g}$ constructed using the curvature tensor and its derivatives up to any finite order are bounded on $\mathscr{M}$.

5. $(\mathscr{M}, \mathfrak{g})$ admits a globally hyperbolic (in the sense of manifolds with boundary) conformal completion with a timelike $\mathscr{I}$. The completion is smooth if n is even, and is of differentiability class at least C^{n-2} if n is odd.

6. (Σ, g_Σ) is a conformally compactifiable manifold, with the same differentiabilities as in point 5.

7. The connected component of the group of isometries of $(\mathscr{M}, \mathfrak{g})$ is exactly $\mathbb{R}$, with an associated Killing vector X being timelike throughout $\mathscr{M}$.

8. There exist no local solutions of the Killing equation other than the (globally defined) timelike Killing vector field X.

An example of a manifold satisfying points 1-6 above is of course n-dimensional anti-de Sitter solution. Clearly it *does not* satisfy points 7 and 8.

We expect that there exist stationary *and not static* solutions as above, which can be constructed by solving an asymptotic Dirichlet problem for the Einstein equations in a conformally compactifiable setting. We are planning to study this question in the future.

In a black hole context we have an obvious variation of the above; we discuss this in more detail in Section 2.3.

Throughout this work we restrict attention to dimension $n \geq 4$.

Our approach is, in some sense, opposite to that in [25,28], where techniques previously used in general relativity have been employed to obtain uniqueness results in a Riemannian setting. Here we start with Riemannian Einstein metrics and obtain Lorentzian ones by "Wick rotation", as follows: Suppose that (M, g) is an n-dimensional conformally compactifiable Einstein manifold of the form $M = \Sigma \times S^1$, and that S^1 acts on M by rotations of the S^1 factor while preserving the metric. Denote by $X = \partial_\tau$ the associated Killing vector field, and assume that X is orthogonal to the sets $\Sigma \times \{\exp(i\tau)\}$, where $\exp(i\tau) \in S^1 \subset \mathbb{C}$. Then the metric g can be (globally) written in the form

$$g = u^2 d\tau^2 + g_\Sigma \ , \qquad \mathscr{L}_X u = \mathscr{L}_X g_\Sigma = g_\Sigma(X, \cdot) = 0 \ . \tag{1.1}$$

It is straightforward to check that the space-time $(\mathscr{M} := \mathbb{R} \times \Sigma, \mathfrak{g})$, with

$$\mathfrak{g} = -u^2 dt^2 + g_\Sigma \tag{1.2}$$

is a static solution of the vacuum Einstein equations with negative cosmological constant, with Killing vector field ∂_t.

In order to continue, some definitions are in order: Let M be the interior of a smooth, compact, n-dimensional manifold-with-boundary $\overline{M}$. A Riemannian manifold (M, g) will be said to be *conformally compact at infinity*, or *conformally compact*, if

$$g = x^{-2} \bar{g} \ ,$$

for a smooth function x on $\overline{M}$ such that x vanishes precisely on the boundary $\partial\overline{M}$ of $\overline{M}$, with non-vanishing gradient there. Further $\bar{g}$ is a Riemannian metric which is regular up-to-boundary on $\overline{M}$; the differentiability properties near ∂M of a conformally compact metric g will always refer to those of $\bar{g}$. The operator

$$P := \Delta_{\mathrm{L}} + 2(n-1) \ ,$$

where Δ_{L} is the Lichnerowicz Laplacian (*cf., e.g.,* [23]) associated with g, plays an important role in the study of such metrics. An Einstein metric g with scalar curvature $-n(n-1)$ will be said *non-degenerate* if P has trivial L^2 kernel on the space of trace-free symmetric 2-tensors. We prove the following openness theorem around static metrics for which the Killing vector field has no zeros (see Section 3.1 for terminology):

THEOREM 1.1 *Let (M,g) be a non-degenerate, strictly globally static conformally compact Riemannian Einstein metric, with conformal infinity $\gamma := [\bar{g}|_{\partial M}]$ (conformal equivalence class). Then any small static perturbation of γ is the conformal infinity of a strictly globally static Riemannian Einstein metric on M.*

Theorem 1.1 is established by the arguments presented at the beginning of Section 2.2, compare [3] for a more detailed treatment. In Section 2.2 we also describe a subclass of Riemannian metrics given by Theorem 1.1 that leads to Lorentzian Einstein metrics with the properties 1-8 listed above. In particular we show that our construction provides non-trivial solutions near the AdS solution in all dimensions.

By an abuse of terminology, Riemannian solutions for which the set of zeros of the Killing vector field X contains an $n-2$ dimensional surface $N \equiv N^{n-2}$ will be referred to as *black hole solutions*; N will be called[2] *the horizon.* The corresponding openness result here reads:

THEOREM 1.2 *Let (M,g) be a non-degenerate, globally static conformally compact Riemannian Einstein metric, with strictly globally static conformal infinity γ. Suppose that*

1. *either $M = N \times \mathbb{R}^2$, with the action of S^1 being by rotations of $\mathbb{R}^2$, or*
2. *$H_{n-3}(M) = \{0\}$, and the zero set of X is a smooth $(n-2)$–dimensional submanifold with trivial normal bundle.*

Then any small globally static perturbation of γ is the conformal infinity of a static black hole solution with horizon N.

The proof of Theorem 1.2 is given at the end of Section 2.3.

This paper is organised as follows: In Section 2.1 we review some results concerning the Riemannian equivalent of the problem at hand. In Section 2.2 we sketch the construction of the new solutions, and we show that our results

[2] In the corresponding Lorentzian solution the set N will be the pointwise equivalent of the *black hole bifurcation surface*, *i.e.*, the intersection of the past and future event horizons.

in the remaining sections prove existence of non-trivial solutions which are near the n-dimensional anti-de Sitter one. In Sections 2.3 and 2.4 we discuss how the method here can be used to produce solutions with black holes, and with Kaluza-Klein type coupling to matter. In Section 3 we give conditions under which hypersurface-orthogonality descends from the boundary to the interior: this is done under the hypothesis of *topological staticity* in Section 3.1, and under the hypothesis of *existence of a twist potential* in Section 3.2. Appendix A studies the action of Killing vector fields near the conformal boundary, and contains results about extendibility of conformal isometries of ∂M to isometries of M. In Appendix B we derive the norm and twist equations for Einstein metrics in all dimensions; those equations are of course well known in dimension four. In Appendix C we study the structure of the metric near fixed points of the action of the isometry group, as needed for the analysis of Section 3. In Appendix D we calculate the sectional curvatures of n-dimensional Kottler-type solutions, proving in particular the existence of a large family of non-degenerate n-dimensional black hole solutions, for any n.

2 The solutions

We start by a review of the associated Riemannian problem:

2.1 The Riemannian solutions

An important result on the structure of conformally compact Einstein manifolds is the following — an improvement of earlier results in [8,21] (*cf.* also [1,2,15] for related or similar results):

THEOREM 2.1 (Lee [23]) *Let M be the interior of a smooth, compact, n-dimensional manifold-with-boundary $\overline{M}$, $n \geq 4$, and let g_0 be a non-degenerate Einstein metric on M that is conformally compact of class $C^{l,\beta}$ with $2 \leq l \leq n-2$ and $0 < \beta < 1$. Let ρ be a smooth defining function for ∂M, and let $\gamma_0 = \rho^2 g_0|_{\partial M}$. Then there is a constant $\epsilon > 0$ such that for any $C^{l,\beta}$ Riemannian metric γ on ∂M with $\|\gamma - \gamma_0\|_{C^{l,\beta}} < \epsilon$, there exists an Einstein metric g on M that has $[\gamma]$ as conformal infinity and is conformally compact of class $C^{l,\beta}$.*

The non-degeneracy condition above will hold *e.g.* in the following circumstances:

THEOREM 2.2 *Under the remaining hypotheses of Theorem 2.1, $\Delta_{\mathrm{L}} + 2(n-1)$ has trivial L^2 kernel on the space of trace-free symmetric 2-tensors if either of the following hypotheses is satisfied:*

(a) At each point, either all the sectional curvatures of g_0 are nonpositive, or all are bounded below by $-2(n-1)/n$.

(b) The Yamabe invariant of $[\gamma]$ is nonnegative and g_0 has sectional curvatures bounded above by $(n-1)(n-9)/8(n-2)$.

Here g_0 has been normalised so that its scalar curvature equals $-n(n-1)$.

Point (b) of Theorem 2.2 is due to Lee [23]. Point (a) is easily inferred from Lee's arguments, for completeness we give the proof in Appendix D.

The regularity of the solutions above can be improved as follows [12]; the following result is the only exception to the rule that $n \geq 4$ in this paper:

THEOREM 2.3 *Let g be a C^2 conformally compactifiable Einstein metric on an n-dimensional manifold M with C^∞ smooth boundary metric $[\gamma]$, $n \geq 3$.*

1. *If n is even or equal to three, then there exists a differentiable structure on $\overline{M}$ such that g is smoothly compactifiable.*

2. *If n is odd, then there exist local coordinates (x, v^C) near the boundary so that*
$$g = x^{-2}(dx^2 + \gamma_{AB}dv^A dv^B) \ , \tag{2.1}$$
with the functions γ_{AB} of the form
$$\gamma_{AB}(x, v^C) = \phi_{AB}(x, v^C, x^{n-1} \ln x) \ , \tag{2.2}$$
with $\phi_{AB}(x, v^C, z)$ — smooth functions of all their arguments. Further, there exists a differentiable structure on $\overline{M}$ so that g is smoothly compactifiable if and only if $(\partial_z \phi_{AB})(0, v^C, 0)$ vanishes.

Explicit formulae for $(\partial_z \phi_{AB})(0, v^C, 0)$ in low dimensions can be found in [22]. It follows from the results in [16] that $(\partial_z \phi_{AB})(0, v^C, 0) = 0$ when $\gamma(0)$ has a representative which is Einstein, so that the filling metric g is smoothly compactifiable in this case, independently of the dimension.

2.2 From Riemannian to Lorentzian solutions

In order to implement the procedure leading from (1.1) to (1.2), we start by solving the Einstein equation for g with a prescribed conformal infinity $[\gamma]$ on $\partial M = \partial\Sigma \times S^1$, using *e.g.* Theorem 2.1. One further assumes that rotations of the S^1 factor are conformal isometries of $[\gamma]$. In order to carry through the construction one needs to know that conformal isometries of $[\gamma]$ extend to isometries of g. This is proved in [1] in dimension four without further restrictions, and under a non-degeneracy condition in higher dimensions; we give an alternative proof of this fact in Appendix A. A somewhat weaker version of the extension result in Appendix A has been independently proved, using essentially the same argument, by Wang [29]; compare [27] for yet another independent similar result. We include the details of our proof, because in the course thereof we derive some properties of Killing vector fields which are used elsewhere in the paper. The final step is to ensure hypersurface-orthogonality of the $U(1)$ action. This is done in the next section.

We refer the reader to [3] for a detailed analysis of the four-dimensional case, where considerably stronger results are available. However, even in dimension four the results on hypersurface-orthogonality in Section 3.2 do not follow from those in [3].

In order to show that the intersection of the set of hypotheses of our results below is not empty, let (M, g_0) be the Riemannian equivalent of the n-dimensional anti-de Sitter space-time, with g_0 obtained by reversing the procedure described above. Thus, M is diffeomorphic to $B^{n-1} \times S^1$, where B^{n-1} is the $(n-1)$–dimensional open unit ball. Let α be any strictly positive function on the unit $(n-2)$–dimensional sphere S^{n-2}, and consider the following metric γ on $S^{n-2} \times S^1$:

$$\gamma = \alpha^2 d\varphi^2 + h \ , \qquad \mathscr{L}_{\partial_\phi} h = h(\partial_\phi, \cdot) = 0 \ . \tag{2.3}$$

Here φ is the coordinate along the S^1 factor of ∂M. Since g_0 has negative sectional curvatures, by Lee's theorem 2.2 there exists an Einstein metric on M with conformal infinity $[\gamma]$ provided that α is sufficiently close to one and h is sufficiently close to the unit round metric.[3] By Proposition A.1 ∂_φ extends to a Killing vector field X on M. Since the corresponding Killing vector field in (M, g_0) did not have any zeros, continuous dependence of solutions of (A.14) upon the metric implies that the same will hold for X (making α closer to 1 and h closer to the unit round metric if necessary). In fact, this also shows that the orbit space M/S^1 will be a smooth manifold, diffeomorphic to B^{n-1}. The fact that $H_{n-3}(B^{n-1}) = \{0\}$ implies existence of the twist potential τ, and since the boundary action is hypersurface orthogonal we can use Theorem 3.3 to obtain hypersurface-orthogonality throughout M. Therefore the Riemannian solutions so obtained lead to Lorentzian equivalents, as described above.

Let us justify the claims made in the Introduction. Point 1 follows immediately from the discussion around Equation (1.1). Point 6 follows from Theorem 2.3 and from what is said in the proof of Theorem 3.3. Point 2 is a straightforward corollary of point 6. Point 4 follows from well known properties of conformally compactifiable metrics. The geodesic completeness of the static metrics so obtained has been proved in [3, Section 4]. Global hyperbolicity in point 5 is established in the course of the proof of Theorem 4.1 of [13], while the differentiability properties claimed in point 5 follow from point 6. The argument given at the end of [3, Section 4] gives non-existence of other global or local Killing vector fields when the boundary metric has no other conformal isometries.

2.3 Black hole solutions

Let us pass to a discussion of static black hole solutions in higher dimensions. The standard examples of static Riemannian AdS-type black hole solutions are on the manifold $M = N^{n-2} \times \mathbb{R}^2$, with $N := N^{n-2}$ compact, and with metric of form:

$$g_m = V^{-1} dr^2 + V d\theta^2 + r^2 g_N \ , \tag{2.4}$$

[3]For definiteness we only consider metrics close to anti-de Sitter, though an identical argument can be used whenever the results of [1], or those of [23], apply. In particular, in view of the results of [1], in dimension four the construction described here applies in much larger generality.

where g_N is any Einstein metric, $Ric_{g_N} = \lambda g_N$, with g_N scaled so that $\lambda = \pm(n-3)$ or 0. Then for $V = V(r)$ given by

$$V = c + r^2 - (2m)/r^{n-3} \ , \tag{2.5}$$

with $c = \pm 1$ or 0 respectively, g_m is static Einstein, with $Ric_{g_m} = -(n-1)g_m$. These are just the analogues of Kottler metrics in higher dimensions. The length of $S^1 \ni \theta$ is determined by m together with the requirement that g_m be a smooth metric at the "horizon" $r = r_0$, the largest root of $V(r)$. (This restriction of course disappears in the Lorentzian setting).

Now each such g_m belongs to a 1-parameter family of metrics, parameterised by the mass m. One expects that for generic values of m, the metric g_m is non-degenerate, as defined in the introduction. In fact, when $n = 4$, the AdS-Schwarzschild metric has non-trivial kernel exactly for one specific value of m,[4] while the toroidal black holes, as well as the higher genus Kottler black holes, are always non-degenerate. Those last two results are well known, and in any case are proved in Appendix D, where we also show existence of a large class of non-degenerate black hole solutions for all $n \geq 5$.

Assuming non-degeneracy, suppose we then consider local perturbations of the conformal infinity, preserving the static structure at infinity — so, *e.g.*, just vary the function, say α, which describes the length of the S^1's at infinity, keeping the remainder of the conformal boundary metric fixed. Then by the results in Appendix A, we get extension of the isometric S^1 action on the (locally unique) Einstein filling metric of Theorem 2.1. We need to prove the extension is static also, this will follow if we can use Theorems 3.1 or 3.4. For the former, we can use the fact that *topological staticity*, as defined at the beginning of Section 3.1, is stable under continuous deformations of the metric (compare [3, Lemma 2.6]; the restriction $\dim M = 4$ there is not needed). This proves Theorem 1.2 under the hypothesis that the action of S^1 is by rotations of $\mathbb{R}^2$, as that action is topologically static.

Another condition which can lead to staticity is $H_{n-3}(M) = 0$, for then we have existence of the twist potential. After a small change of γ, the S^1 action is again a small perturbation of the original static S^1 action. This means that the only fixed point set of the S^1 action (the zero set of X), is again a smooth $(n-2)$ manifold, say N, with the normal S^1 bundle remaining trivial. Hence, condition (3.20) of Theorem 3.4 is satisfied, and staticity follows.

2.4 Kaluza-Klein solutions

Similarly one should be able to construct space-times as described in the introduction, with or without black hole regions, which satisfy the Einstein-Yang-Mills-dilaton field equations with a Kaluza-Klein coupling [14]; solutions belonging to this family have been numerically constructed in [9, 30]. More precisely, suppose that one has a conformally compactifiable Einstein manifold (M^n, g)

[4] In dimension four it is known that all continuous isometries descend from the boundary to the interior [1]. When infinity is spherical, the elements of the kernel have to be spherically symmetric, and one can conclude using the (generalised) Birkhoff theorem.

satisfying the following: a) $\partial M^n = S^1 \times M^{n-2}$; with a product boundary conformal class $[\tilde{g}|_{\partial M^n}]$ such that b) rotations along S^1 are conformal isometries; c) (M^n, g) satisfies the hypotheses of Theorem 2.1 (compare Theorem 2.2); d) the S^1 action on M associated with the rotations of S^1 on ∂M satisfies the hypotheses of one of the Theorems 3.1, 3.3, 3.4 or 3.6. Then any connected Lie group G of conformal isometries of $[\tilde{g}|_{M^{n-2}}]$ with a free action will then lead to a Kaluza-Klein type Yang-Mills gauge group for the associated Lorentzian solutions.

3 Hypersurface-orthogonality

We use the notations of Appendix A. We wish to show that X is hypersurface-orthogonal; clearly a necessary condition for that is that $\hat{X}(0)$ be hypersurface orthogonal. We suspect that this condition is sufficient, but we have not been able to prove that. In dimension four hypersurface-orthogonality has been proved in [3] under the hypothesis of *topological staticity* of X, as defined there, *cf.* below. We shall show in Section 3.1 that the result generalises to higher dimensions. We also give an alternative approach, assuming that we have the *twist potential* at our disposal.

In what follows we will assume that $\hat{X}(0)$ arises from an S^1 action on ∂M; this hypothesis can be replaced by the existence of a hypersurface $\mathscr{S}$ in $M \setminus \{g(X,X)=0\}$ which is transversal to X — identical proofs apply, with Σ there replaced by $\mathscr{S}$.

Before proceeding further we have to introduce some notation. Let Σ be the orbit space of the S^1 action, and let $\mathring{\Sigma}$ denote the set of orbits of principal type, then $\mathring{\Sigma}$ is a smooth manifold forming an open dense subset of Σ. We set

$$\mathring{M} := (\pi^*_\Sigma)^{-1}\mathring{\Sigma} \; .$$

Let g_Σ be the induced metric on $\mathring{\Sigma}$; thus the metric g has the form

$$g = u^2(d\phi + \theta)^2 + \pi^*_\Sigma g_\Sigma, \tag{3.1}$$

where θ is a connection 1-form, u is the length of the Killing field $X = \partial/\partial\phi$, and $\pi^*_\Sigma : M \to \Sigma$ is the canonical projection. The parameter ϕ parameterises a circle S^1. The space Σ is in general a $(n-1)$-orbifold; there may be stratified submanifolds in Σ along which the metric has cone singularities. Of course Σ is non-compact — it has a boundary at infinity $\partial_\infty\Sigma$ corresponding to the orbit space of the S^1 action on ∂M. Redefining the S^1 action if necessary, one can without loss of generality assume that the action is free on $(\pi^*_\Sigma)^{-1}\mathring{\Sigma}$. The set of non-principal orbits is the union of trivial orbits $\Sigma_{\rm sing,0}$ which correspond to fixed points of the action, and of special orbits $\Sigma_{\rm sing,iso}$ which are circles with non-trivial isotropy group:

$$\Sigma_{\rm sing} := \Sigma_{\rm sing,0} \cup \Sigma_{\rm sing,iso} \; , \qquad M_0 := (\pi^*_\Sigma)^{-1}\Sigma_{\rm sing,0} = \{u=0\} \subset M \; .$$

In dimension four the fixed point set consists of isolated points and smooth, totally geodesic submanifolds, those have been called "nuts" and "bolts"; nut

fixed points are isolated points in M, while the bolts correspond to totally geodesic surfaces in $\partial\mathring{\Sigma}$, *cf.* [20]. The structure of the orbits near fixed points is discussed in all dimensions in Appendix C, see also [17,18]. Off the fixed point set the isotropy group is finite and so the orbits are circles. The isotropy group may change. For example, in dimension three one can have S^1 actions on a solid torus $D^2 \times S^1$ which are free on $(D^2 \setminus \{0\}) \times S^1$, with non-trivial isotropy on the core curve $\{0\} \times S^1$. One can take such and product with S^1 again to obtain higher dimensional manifolds which have (isolated) curves where the isotropy jumps up [10,17,18].

Let the *twist* $(n-3)$*–form* $\hat\omega$ be defined on M by the equation (the definition here differs by a factor of 2 from the definition in [3])

$$\hat\omega = *_g(\xi \wedge d\xi) \ , \qquad \xi := g(X,\cdot) \ ,$$

where $*_g$ is the Hodge duality operator with respect to the metric g. On $\mathring{M}$ the form $\hat\omega$ is the pull-back by π_Σ of a form ω defined on $\mathring{\Sigma}$. It is well known in dimension four, and it is shown in general in Appendix B, that ω (and hence $\hat\omega$) is closed.

3.1 Topologically static actions

We will use the following terminology, as in [3]. The S^1 action on (M,g) is *strictly globally static* if (M,g) is globally a warped product of the form

$$M = S^1 \times \Sigma \ , \quad g = u^2 d\phi^2 + \pi_\Sigma^* g_\Sigma \ , \tag{3.2}$$

where $u : \Sigma \to \mathbb{R}$ is strictly positive and g_Σ is a complete metric on Σ, $\partial\Sigma = \emptyset$. In this case, the S^1 action is just given by rotations in the S^1 factor. The S^1 action is *globally static* if (3.2) holds with $u = 0$ somewhere. In this case, the locus $\{u = 0\}$ is not empty, but there are no exceptional orbits. Next, the S^1 action is *topologically static* if the S^1 bundle $S^1 \to P \to \Sigma_P$ is a trivial bundle, i.e. it admits a section. (Here P is the union of principal orbits, while we use the symbol E for the union of exceptional ones.) This is equivalent to the existence of a cross-section of the S^1 fibration $P \cup E \to \Sigma_{P\cup E}$. [5] Finally, we define the S^1 action to be *locally static* if every point of (M,g) has a neighborhood isometric to a neighborhood of a point with metric of the form (3.2); this is equivalent to the usual notion of static in the sense of the existence of a hypersurface orthogonal Killing field.

We shall use an obvious equivalent of the above for an $\mathbb{R}$ action by isometries on a Lorentzian manifold $(\mathscr{M}, \mathfrak{g})$, with the further restriction that the associated Killing vector field be timelike almost everywhere.

The main result of this section is the following:

THEOREM 3.1 *Let (M,g) be a smoothly compactifiable Einstein metric on M, with $\dim M \geq 4$. Suppose the free S^1 action at conformal infinity $(\partial M,\gamma)$ is strictly globally static, i.e.*

$$\partial M = S^1 \times V, \tag{3.3}$$

[5] S will be called a cross-section of a fibration if S meets every fiber at least once, with the intersection being transverse.

and the S^1 action on (M,g) is topologically static.

Then the S^1 action on (M,g) is locally static*, i.e. (M,g) is locally of the form* (3.2)*, (with $\{u=0\}\neq\emptyset$ possibly).*

PROOF: The method of proof is identical to that in [3]. As pointed out in [27], regardless of dimension and signature one has the identity[6]

$$d\left(\frac{\xi\wedge\hat\omega}{u^2}\right)=\pm\frac{|\hat\omega|^2 i_X(\mathrm{Vol})}{u^4}\ , \tag{3.4}$$

with the sign $\pm$ being determined by the signature of the metric. Here Vol is the volume form. Integrating (3.4) over any cross-section Σ one will obtain $\hat\omega=0$ provided that the boundary term arising from the left-hand-side of (3.4) vanishes. This requires sufficiently fast fall-off of $\hat\omega$ near the conformal infinity ∂M, which is provided by the following Lemma. The hypotheses of Theorem 3.1 imply that the Killing vector has no zeros on ∂M, but we do not need this assumption for the proof that follows:

LEMMA 3.2 *In the coordinate system of* (A.1) *we have*

$$\hat\omega_{rA_1\ldots A_{n-4}}=O(r)\ ,\qquad \hat\omega_{AA_1\ldots A_{n-4}}=O(1)\ ,$$

with $\hat\omega_{rA_1\ldots A_{n-4}}=\omega_{rA_1\ldots A_{n-4}}$ and $\hat\omega_{AA_1\ldots A_{n-4}}=\omega_{AA_1\ldots A_{n-4}}$.

PROOF: The idea of the proof is to use the equation

$$\nabla^k\nabla_k X_i=-\,\mathrm{Ric}(g)_i{}^jX_j\ ,$$

together with the fact that g is Einstein, to obtain more information about the decay of the relevant components of the metric. We work in the coordinate system of Appendix A, and use the conventions there. We have (recall that $\hat X_r\equiv 0$ by Remark A.3)

$$\begin{aligned}
2\nabla^k\nabla_kX_A &= \nabla^k(\nabla_kX_A-\nabla_AX_k)\\
&= \nabla^k(\partial_kX_A-\partial_AX_k)\\
&= r^2\Big\{\partial_r^2X_A-\Gamma^r_{rr}\partial_rX_A-\Gamma^E_{rA}\partial_rX_E+\hat g^{EF}\Big[\partial_E\partial_FX_A-\partial_E\partial_AX_F\\
&\qquad -\Gamma^r_{EF}\partial_rX_A-\Gamma^C_{EF}(\partial_CX_A-\partial_AX_C)\\
&\qquad +\Gamma^r_{EA}\partial_rX_F-\Gamma^C_{EA}(\partial_FX_C-\partial_CX_F)\Big]\Big\}\ .
\end{aligned}\tag{3.5}$$

Consider any point p on the conformal boundary at which $\hat X(0)$ does not vanish. As $\hat X(0)$ is hypersurface orthogonal, we can choose a local coordinate system on the boundary at infinity, defined on a neighborhood of p, such that $x^A=(\varphi,x^a)$ $(a=3,...,n)$, with $\hat X(0)=\partial_\varphi$ and

$$\hat g(0)_{AB}dx^Adx^B=\hat g(0)_{\varphi\varphi}d\varphi^2+\hat g(0)_{ab}dx^adx^b.$$

[6] The identity here differs by a factor of 2 from the one in [3] because the form ω here is twice that in [3].

Recall that $X_A = r^{-2}\hat{g}_{AB}\hat{X}^B = r^{-2}\hat{g}_{A\varphi}$. From (A.2)–(A.3) and (3.5) we then have

$$\begin{aligned}2\nabla^k\nabla_k X_A &= 2(n-1)r^{-2}\hat{g}_{A\varphi} - nr^{-1}\hat{g}'_{A\varphi} + r^{-1}\hat{g}^{CD}(2\hat{g}'_{DA}\hat{g}_{C\varphi} - \hat{g}'_{CD}\hat{g}_{A\varphi})\\ &\quad +\hat{g}''_{A\varphi} + \tfrac{1}{2}\hat{g}^{CD}(-2\hat{g}'_{DA}\hat{g}'_{C\varphi} + \hat{g}'_{CD}\hat{g}'_{A\varphi}) + 2\hat{\nabla}^E\hat{\nabla}_E\hat{X}_A\ ,\end{aligned}$$

where a prime denotes an r-derivative. From equation (A.10) we have

$$\hat{g}'_{A\varphi} = \hat{g}^{CD}\hat{g}'_{DA}\hat{g}_{C\varphi}$$

and then as $\mathrm{Ric}(g) = -(n-1)g$,

$$\begin{aligned}&(2-n)r^{-1}\hat{g}'_{A\varphi} + \hat{g}''_{A\varphi}\\ &\qquad = r^{-1}\hat{g}^{CD}\hat{g}'_{CD}\hat{g}_{A\varphi} - \tfrac{1}{2}\hat{g}^{CD}(-2\hat{g}'_{DA}\hat{g}'_{C\varphi} + \hat{g}'_{CD}\hat{g}'_{A\varphi}) - 2\hat{\nabla}^E\hat{\nabla}_E\hat{X}_A.\end{aligned}$$

Consider that equation when $A = a$; straightforward but tedious algebra shows that its right-hand-side can be written as a linear combination of the $\hat{g}_{b\varphi}$'s together with their first derivatives and their second ∂_c–derivatives, with bounded, differentiable up-to-boundary, coefficients built out of the g_{AB}'s and their derivatives. (For example,

$$\hat{g}^{CD}\hat{g}'_{Da}\hat{g}'_{C\varphi} \quad = \quad \hat{g}^{\varphi\varphi}\hat{g}'_{\varphi a}\hat{g}'_{\varphi\varphi} + \hat{g}^{c\varphi}\hat{g}'_{\varphi a}\hat{g}'_{c\varphi} + \hat{g}^{c\varphi}\hat{g}'_{ca}\hat{g}'_{\varphi\varphi} + \hat{g}^{cd}\hat{g}'_{da}\hat{g}'_{d\varphi}\ ,$$

and note that $\hat{g}^{c\varphi}$ is a rational function involving the $g_{b\varphi}$'s which vanishes when the latter do, hence can be written as an expression linear in the $g_{b\varphi}$'s with coefficients which depend upon the g_{AB}'s.) Then as $\hat{g}_{a\varphi}(0) = 0$ we have that $\hat{g}_{a\varphi} = O(r^2)$. Taylor expanding and matching coefficients in front of powers of r one is led to

$$\hat{g}_{a\varphi} = O(r^{n-1}). \tag{3.6}$$

Set

$$\mu^2 := \hat{g}(0)(\hat{X}(0), \hat{X}(0))\ .$$

Since g is Einstein we have [16] $\hat{g}(r) = \hat{g}(0) + O(r^2)$, so that

$$u^2 := g(X,X) = r^{-2}\hat{g}(\hat{X},\hat{X}) = r^{-2}(\mu^2 + O(r^2))\ . \tag{3.7}$$

In particular u behaves as $1/r$ (recall that we are so far working away from the zero set of $\hat{X}(0)$). Consider the two-form $\hat{\lambda}$ defined in (B.9); (3.6) gives

$$\hat{\lambda} = \sum_a O(r^{n-5})dr\wedge dx^a + \sum_{a,b} O(r^{n-4})dx^a\wedge dx^b\ . \tag{3.8}$$

The coordinates (r, x^a) can be used as local coordinates on the quotient manifold Σ, in those coordinates $\hat{\lambda}_{ab} = \lambda_{ab}$, $\hat{\lambda}_{ar} = \lambda_{ar}$, $\hat{\lambda}_{i\varphi} = 0$. It follows now that in any coordinate system $\{x^A\}$ on the conformal boundary as in (A.1), not necessarily adapted to the hypersurface-orthogonal character of X, we will have

$$\lambda_{AB} = O(r^{n-4})\ , \qquad \lambda_{Ar} = O(r^{n-5})\ , \tag{3.9}$$

as long as we are not at a point at which $\hat{X}(0)$ vanishes. Now, $u\hat{\lambda}$ is defined and smooth regardless of zeros of X, which implies that (3.9) holds globally on

each domain of definition of the coordinates x^A, independently of the existence of zeros of $\hat{X}(0)$ there. We finally obtain

$$\lambda_{AB} \quad = \quad O(r^{n-4}) \quad \Longleftrightarrow \quad \lambda^{AB} = g^{AC} g^{BD} \lambda_{CD} = O(r^n) \ , \tag{3.10}$$

with a similar equivalence for λ_{Ar}. Let $\eta_{A_1 \ldots A_{n-2}}$ be totally anti-symmetric, equal to 1 if $A_1 \ldots A_{n-2}$ is an even permutation of $1, 2, \ldots, n-2$. It is clear from (A.12)-(A.13) that

$$\sqrt{\det g_\Sigma} = r^{-(n-1)} \sqrt{\det \hat{g}_\Sigma(r)} \ ,$$

where $\hat{g}_\Sigma(r)$ is the metric on the quotient $(\{r = \mathrm{const}\}, \hat{g}(r))/S^1$. Choosing a convenient orientation, from the definition (B.16) of ω we have

$$\begin{aligned} \omega_{rA_1 \ldots A_{n-4}} &= \sqrt{\det g_\Sigma}\, \eta_{A_1 \ldots A_{n-4} BC} \lambda^{BC} \\ &= r^{-(n-1)} \sqrt{\det \hat{g}_\Sigma(r)}\, \eta_{A_1 \ldots A_{n-4} BC} \lambda^{BC} \\ &= O(r) \ , \end{aligned} \tag{3.11}$$

as desired. The claim about $\omega_{AA_1 \ldots A_{n-4}}$ is established by a similar calculation. □

Returning to the proof of Theorem 3.1, suppose first that the Killing vector field X has no zeros, and that all orbits are of principal type. Let S be a hypersurface transverse to X, let $S(r)$ denote the intersection of S with the level sets of the function r of (A.1), we then have

$$\int_S \frac{1}{u^4} |\hat{\omega}|^2 * \xi = \pm \lim_{r \to 0} \int_{S(r)} \frac{1}{u^2} \xi \wedge \hat{\omega} \ , \tag{3.12}$$

with the ± 1 factor as in (3.4). In local coordinates of (A.1) we have $\xi = g(X, .) = O(r^{-2})$ and $u^2 \geq c r^{-2}$, while $\hat{\omega} = O(1)$ by Lemma 3.2. Further, if we choose S to be asymptotically orthogonal to X, then the pull-back of ξ to S will be $o(r^{-2})$. Thus we obtain

$$\int_S \frac{1}{u^4} |\hat{\omega}|^2 * \zeta = \lim_{r \to 0} o(1) = 0.$$

Now $*\xi = \alpha |K| dvol_S$, where α, the angle between S and K, is of constant sign. We can conclude $\hat{\omega} = 0$.

When zeros of X occur, (3.12) becomes

$$\int_S \frac{1}{u^4} |\hat{\omega}|^2 * \xi = \pm \lim_{r \to 0} \int_{S(r)} \frac{1}{u^2} \xi \wedge \hat{\omega} \mp \lim_{\epsilon \to 0} \int_{\{\rho = \epsilon\}} \frac{1}{u^2} \xi \wedge \hat{\omega} \ . \tag{3.13}$$

Here $\rho = \sqrt{\sum_{i=1}^{\ell} \rho_i^2}$, with ρ_i as in (C.12). To finish the proof, we need to show that the boundary integral corresponding to the zeros of X vanishes. We use the coordinate system of (C.8): we have $c^{-1} \rho \leq u \leq c \rho$ by (C.26). Then $\hat{\omega} = O(\rho)$ by (C.28) and $\xi = O(\rho)$ by (C.15), hence the integrand in the right-hand-side of (3.13) is uniformly bounded as $\epsilon \to 0$. By scaling (or by a direct calculation,

using the formulae of Appendix C) one sees then that the integral vanishes at least as fast as $\epsilon^{2\ell-1}$, whence the result. $\square$

The above result is reasonably satisfactory from a general relativistic point of view: in that case the solutions of main interest possess spacelike hypersurfaces transverse to the Killing vector field, which imply topological staticity of the associated Riemannian solution. Nevertheless, it seems of interest to look for other hypotheses which will lead to hypersurface-orthogonality of the Killing vector. In the next section we will obtain some such results under the hypothesis that there exists a *twist potential* τ, *i.e.*, $\omega = d\tau$.

3.2 Solutions with a twist potential

Our next result assumes that ω is exact and that X has no zeros. The case with zeros will be covered in Theorems 3.4 and 3.6, while the question of exactness of ω will be addressed in Theorem 3.7; notations and conventions of Appendix B are used.

In the result that follows we assume that $(\partial M, \gamma)$ is not conformal to the round sphere. That last case is covered by [5] when M is spin, and by [1] or [27], (together with [11]), regardless of the existence of a spin structure; in those works it is shown that (M, g) is then the hyperbolic space. In our context a simple proof can be given assuming non-degeneracy, for then every continuous isometry descends to the interior, and the result follows by ODE methods.

THEOREM 3.3 *Let (M, g) be Einstein, assume that $(\partial M, \gamma)$ is not conformal to the round sphere, and suppose that $\hat{X}(0)$ is hypersurface-orthogonal on ∂M. Assume further that there exists on $\mathring{\Sigma}$ a $(n-4)$–form τ such that*

$$\omega = d\tau \, . \tag{3.14}$$

If X has no zeros, then X is hypersurface-orthogonal on M.

Both the $(n-4)$–form τ of (3.14), as well as its M–counterpart $\hat{\tau} = \pi_\Sigma^* \tau$, will be referred to as *the twist potentials.*

PROOF: We use the notation of Appendix B throughout. Let r be the coordinate of (A.1). By Remark A.3 the function r passes to the quotient $\Sigma = M/S^1$, and by an abuse of notation we shall use the same letter for the resulting function. For $\rho > 0$ set

$$\Sigma(\rho) = \Sigma \setminus (\{r < \rho\} \cup \{p : d(p, \Sigma_{\rm sing}) < \rho\}) \subset \mathring{\Sigma} \, .$$

By (B.20) we have

$$d(u^{-3} *_{g_\Sigma} \omega) = 0 \, . \tag{3.15}$$

Taking the exterior product of this equation with τ and integrating over $\Sigma(\rho)$

one has

$$
\begin{aligned}
0 &= (-1)^{(n-4)} \int_{\Sigma(\rho)} \tau \wedge d(u^{-3} *_{g_\Sigma} \omega) \\
&= \int_{\partial\Sigma(\rho)} u^{-3}\tau \wedge *_{g_\Sigma}\omega - \int_{\Sigma(\rho)} u^{-3} d\tau \wedge *_{g_\Sigma}\omega \\
&= \int_{\partial\Sigma(\rho)} u^{-3}\tau \wedge *_{g_\Sigma}\omega - \int_{\Sigma(\rho)} u^{-3} |\omega|^2_{g_\Sigma} *_{g_\Sigma} 1 \, . \qquad (3.16)
\end{aligned}
$$

The idea is to show that the boundary integral above vanishes when passing with ρ to zero, yielding $\omega = 0$, as desired.

For ρ small enough $\partial\Sigma(\rho)$ is a finite union of smooth submanifolds of $\mathring{\Sigma}$ of co-dimension one. The simplest case is $\mathring{\Sigma} = \Sigma$, this occurs when X has no zeros and all orbits are of principal type, so that $\Sigma_{\mathrm{sing}} = \emptyset$ and $\partial\Sigma(\rho)$ equals

$$\partial_\infty \Sigma(\rho) := \{r = \rho\} \, .$$

In general $\partial\Sigma(\rho)$ might have further components of the form

$$\partial\Sigma_{\mathrm{sing,iso}}(\rho) := \{p : d(p, \Sigma_{\mathrm{sing,iso}}) = \rho\}$$

and also

$$\partial\Sigma_{\mathrm{sing,0}}(\rho) := \{p : d(p, \Sigma_{\mathrm{sing,0}}) = \rho\} \, .$$

The latter are, however, excluded by our current hypothesis that X has no zeros on M.

Lemma 3.2 and the definition (3.14) of τ give

$$
\begin{aligned}
(n-3)\partial_{[r}\tau_{A_1\ldots A_{n-4}]} &= \omega_{rA_1\ldots A_{n-4}} \\
&= O(r) \, , \qquad (3.17)
\end{aligned}
$$

where square brackets over a set of indices denote complete anti-symmetrisation with an appropriate combinatorial factor ($1/((n-3)!)$ in the current case). In dimension four this gives

$$\partial_r \tau = O(r) \, ,$$

while in higher dimensions one obtains

$$\partial_{[r}\tau_{A_1\ldots A_{n-4}]} = \partial_r \tau_{A_1\ldots A_{n-4}} + (-1)^{n-4} \partial_{[A_1}\tau_{A_2\ldots A_{n-4}]r} = O(r) \, .$$

By integration we are led to

$$\tau_{A_1\ldots A_{n-4}} = \sigma_{A_1\ldots A_{n-4}} + O(1) \, , \qquad (3.18)$$

where $\sigma_{A_1\ldots A_{n-4}} = 0$ in dimension four, and

$$\sigma_{A_1\ldots A_{n-4}} := -\int_{r_0}^{r} (-1)^{n-4} \partial_{[A_1}\tau_{A_2\ldots A_{n-4}]r} dr$$

otherwise. Let us use the symbol $\tilde{d}$ to denote the exterior differential on $\partial_\infty\Sigma(\rho)$, at fixed ρ. Then

$$\begin{aligned}
\sigma &:= \frac{1}{(n-4)!}\sigma_{A_1\ldots A_{n-4}}dx^{A_1}\wedge\ldots\wedge dx^{A_{n-4}} \\
&= \tilde{d}\left[-\frac{1}{(n-4)!}\left(\int_{r_0}^{r}(-1)^{n-4}\tau_{A_2\ldots A_{n-4}r}dr\right)dx^{A_2}\wedge\ldots\wedge dx^{A_{n-4}}\right] =: \tilde{d}\tilde{\sigma}\ .
\end{aligned}$$

We note that

$$\int_{\partial_\infty\Sigma(\rho)} u^{-3}\tilde{d}\tilde{\sigma}\wedge *_{g_\Sigma}\omega = \int_{\partial_\infty\Sigma(\rho)} u^{-3}d\tilde{\sigma}\wedge *_{g_\Sigma}\omega = \int_{\partial_\infty\Sigma(\rho)}\tilde{\sigma}\wedge d(u^{-3} *_{g_\Sigma}\omega) = 0\ ,$$

so that Equations (3.10) and (3.18) imply

$$\int_{\partial_\infty\Sigma(\rho)} u^{-3}\tau\wedge *_{g_\Sigma}\omega \ = \ \int_{\partial_\infty\Sigma(\rho)} u^{-3}\tau\wedge\lambda = O(\rho^{n-1})\ ,$$

which tends to zero as ρ tends to zero. If $\Sigma = \mathring{\Sigma}$ we are done.

Since we have assumed that X has no zeros, it only remains to analyse the boundary integral around the S^1 orbits with a non-trivial isotropy group. By point 1 of Proposition C.1 below such orbits necessarily form a lower-dimensional subset of Σ, with u being uniformly bounded in a neighborhood thereof. We are thus integrating a bounded $(n-2)$–form over a submanifold, the $(n-2)$–area of which shrinks to zero as ρ tends to zero, which leads to a vanishing contribution in (3.16) in the limit. $\square$

We wish, next, to prove an equivalent of Theorem 3.3 that allows zeros of X. The proof will again proceed via the identity (3.16), except that we will have now a supplementary contribution from $\partial\Sigma_{\mathrm{sing},0}(\rho)$. Let $\hat{F}$ be the curvature of the $U(1)$–principal bundle of unit normals to $M_{0,n-2}$, obtained from the $U(1)$–connection $\gamma_a dx^a$ defined in (C.21); in local coordinates,

$$F := d(\gamma_a dx^a)\ ,\quad \hat{F} := \pi_\Sigma^* F\ . \tag{3.19}$$

We shall use the notation and terminology of Appendix C. We have:

THEOREM 3.4 *Under the remaining hypotheses of Theorem 3.3, assume instead that* $\cup_\ell M_{0,n-2\ell}\neq\emptyset$ *and that*

$$\int_{M_{0,n-2}}\hat{\tau}\wedge\hat{F} - \frac{2\pi}{\kappa_1\kappa_2}\int_{M_{0,n-4}}\hat{\tau} = 0\ . \tag{3.20}$$

Then

$$\cup_{\ell\geq 2}M_{0,n-2\ell} = \emptyset\ , \tag{3.21}$$

and the conclusions of Theorem 3.3 hold.

PROOF: By Proposition C.1 the set $\Sigma_{\mathrm{sing},0}$ is the projection by π_Σ of a disjoint union of smooth, non-intersecting, submanifolds of dimension $n-2\ell$, $0\leq\ell\leq n/2$. It is thus sufficient to consider each such manifold separately. Consider,

then, a connected component of $\partial\Sigma_{\rm sing,0}(\rho)$ which is a projection of a connected component of $M_{0,n-2\ell}$ for some ℓ, and suppose that

$$\partial\Sigma_{\rm sing,iso}(\rho)\cap\partial\Sigma_{\rm sing,0}(\rho)=\emptyset$$

for ρ small enough. (Proposition C.2 shows that this occurs precisely for those components of $\partial\Sigma_{\rm sing,0}$ for which the associated $M_{0,n-2\ell}$'s have all κ_i's equal to one.)

Consider, first, the case $\ell=1$; using (C.23), (B.16) and (B.9) we find (recall that $\hat\lambda$ can be identified with λ in the adapted coordinate system used)

$$\begin{aligned}\lim_{\rho\to0}\int_{\partial\Sigma_{\rm sing,0}(\rho)}u^{-3}\tau\wedge *_{g_\Sigma}\omega &= \int_{\pi_\Sigma(M_{0,n-2})}\tau\wedge F\\ &= \int_{M_{0,n-2}}\hat\tau\wedge\hat F\ .\end{aligned}\tag{3.22}$$

In dimension n equal to four the last term in (3.22) is the value of τ at the connected component of $M_{0,n-2}$ under consideration multiplied by the Euler class of the principal $U(1)$–bundle of unit vectors normal to $M_{0,n-2}$. Regardless of the dimension n, we have:

Proposition 3.5 *When the normal bundle of $M_{0,n-2}$ is trivial the first integral in* (3.20) *vanishes.*

Proof: If the normal bundle of $M_{0,n-2}$ is trivial, then $\gamma_a dx^a$ is defined globally on $M_{0,n-2}$, and the integrand in (3.22) integrates out to zero:

$$(-1)^{n-4}\int_{\pi_\Sigma(M_{0,n-2})}\tau\wedge d(\gamma_a dx^a)=\int_{\pi_\Sigma(M_{0,n-2})}\left(d\left(\tau\wedge\gamma_a dx^a\right)-d\tau\wedge\gamma_a dx^a\right)=0\ ;\tag{3.23}$$

recall that $d\hat\tau=0$ on $M_{0,n-2\ell}$. (Strictly speaking, for $\ell\geq2$ one should do the above calculation on $\partial\Sigma_{\rm sing,0}(\rho)$ and then pass to the limit, since $\partial\Sigma_{\rm sing,0}$ does not have a differentiable manifold structure in general for $\ell\geq2$ — while τ extends by continuity to $M_{0,n-2\ell}$ in the coordinates of Appendix C, the exterior derivative $d\tau$ of τ might not be defined there). □

Returning to the proof of Theorem 3.4 suppose, next, that $\ell\geq2$. In the coordinates $(\rho_1,(\rho_i,\psi_i)_{i=2,\ell-1},x^a)$ of (C.27) the boundary integrand in (3.16) is of order of ρ^{-1} while

$$\partial\Sigma_{\rm sing,iso}(\rho)=\{\rho_i\geq0\ ,\ \rho_1^2+\ldots+\rho_\ell^2=\rho^2\ ,\ \psi_i\in[0,2\pi]\ ,\ x^a\in M_{0,n-2\ell}\}$$

has (coordinate Lebesgue) measure $O(\rho^{\ell-1})$, hence

$$\int_{\partial\Sigma_{\rm sing,0}(\rho)}u^{-3}\tau\wedge *_{g_\Sigma}\omega=O(\rho^{\ell-2})\ .\tag{3.24}$$

The simplest case is then $\ell\geq3$, which immediately gives zero contribution in the limit. Equation (3.24) also shows that the $M_{0,n-2\ell}$'s with $\ell=2$ give a finite contribution as ρ tends to zero. Clearly, the only terms that might give a

non-zero contribution in the limit are those which arise from the second line of (C.27). If the dimension of M is four then the second term there does not occur. The first term looks like a total divergence so one is tempted to conclude that it gives a zero contribution when integrated upon. This is, however, deceptive, because the coordinate system used there is singular at the set $\rho_1 = 0$, and around each connected component of $\pi_\Sigma(M_{0,n-4})$ from (C.27) one finds

$$\begin{aligned}\lim_{\rho\to 0}\int_{\partial\Sigma_{\mathrm{sing},0}(\rho)} u^{-3}\tau\wedge *_{g_\Sigma}\omega &= -\frac{2\pi}{\kappa_1\kappa_2}\int_{\pi_\Sigma(M_{0,n-4})}\tau \\ &= -\frac{2\pi}{\kappa_1\kappa_2}\int_{M_{0,n-4}}\hat\tau\ ; \end{aligned}\tag{3.25}$$

the $1/\kappa_1 = \kappa_1 = \pm 1$ factor arises from a change of orientation. In dimension four each connected component of $M_{0,n-4}$ is a point, and the integral here is understood as the value of $\hat\tau$ at the point under consideration; (3.25) gives the contribution from the first term in the second line of (C.27) for all n. It can be checked that for $n > 4$, the second term there gives a vanishing contribution in the limit, so that (3.25) holds for all dimensions. (Strictly speaking, at this stage $\kappa_2 = 1$ in the formula above, as we have assumed that all the nearby orbits have period equal either 0 or 2π. However, as shown below, the above formula also gives the boundary contribution around the $\pi_\Sigma(M_{0,n-4})$'s in general.)

Clearly (3.25) depends only upon the $\pi_\Sigma(M_{0,n-4})$–homology class of the restriction $\mathring\tau$ of τ to $\pi_\Sigma(M_{0,n-4})$.

Let us show how to reduce the general case to the previous one. As explained in Appendix C, in the coordinate patch $\mathscr{U}_p$ defined there the surface $\partial\Sigma_{\mathrm{sing,iso}}(\rho)$ takes the form

$$\partial\Sigma_{\mathrm{sing,iso}}(\rho)\cap\mathscr{U}_p = \left\{\sum_{i=i_2}^{\ell}\rho_i^2 = \rho\right\}\ .$$

We can deform those surfaces to

$$\{\sum_{i=1}^{\ell}\rho_i^2 = \rho\}$$

using the family of surfaces

$$\partial\Sigma(\rho,\delta) := \underbrace{\left\{\sum_{i=1}^{\ell}\rho_i^2 = \rho, \sum_{i=i_2}^{\ell}\rho_i^2 \ge \delta\right\}}_{\partial\Sigma_1(\rho,\delta)} \cup \underbrace{\left\{\sum_{i=1}^{\ell}\rho_i^2 \ge \rho, \sum_{i=i_2}^{\ell}\rho_i^2 = \delta\right\}}_{\partial\Sigma_2(\rho,\delta)}\ ,$$

with $0\le\delta\le\rho$. At fixed ρ, on $\partial\Sigma(\rho,\delta)$ the integrand is uniformly bounded while $\partial\Sigma(\rho,\delta)$ shrinks to a lower-dimensional object as δ tends to zero, therefore

$$\lim_{\delta\to 0}\int_{\partial\Sigma_2(\rho,\delta)} u^{-3}\tau\wedge *_{g_\Sigma}\omega = 0\ .$$

This reduces the problem of calculating the limit, as ρ goes to zero, of the integral of $u^{-3}\tau\wedge *_{g_\Sigma}\omega$ over $\partial\Sigma_{\mathrm{sing,iso}}(\rho)\cap\mathscr{U}_p$, to that of calculating

$$\lim_{\rho\to 0}\int_{\partial\Sigma_1(\rho,0)} u^{-3}\tau\wedge *_{g_\Sigma}\omega\ .$$

But this is an integral already considered under the assumption that $\partial\Sigma_{\rm sing,iso}$ does not meet $\partial\Sigma_{\rm sing,0}$ in $\mathscr{U}_p$, and an identical analysis applies.

Those components of $\Sigma_{\rm sing,iso}$ which do not meet $\Sigma_{\rm sing,0}$, or which lie away from the $\mathscr{U}_p$'s, are handled as in the proof of Theorem 3.3. Finally, (3.21) is a rephrasing of Proposition C.3. □

There are various ways to ensure that (3.20) holds: Suppose, for instance, that we are in dimension four, then $\hat\tau$ is a function on M, defined up to a constant; further $\hat\tau$ is constant on any connected component of $M_{0,n-2\ell}$. In this case, when $\cup_{\ell\le 2}M_{0,n-2\ell}$ is connected, we can choose τ to be zero on $\cup_{\ell\le 2}M_{0,n-2\ell}$, obtaining a vanishing contribution from $\cup_{\ell\le 2}M_{0,n-2\ell}$. Another possibility is to assume that the bundle of unit normals to $M_{0,n-2}$ is trivial, see Proposition 3.5. If, moreover, $M_{0,n-4}$ is connected (which will certainly be the case if it is empty), then we can choose again the constant of integration appropriately to achieve the desired equality. One can clearly assume various combinations of the hypotheses above. As a special case, we have obtained:

THEOREM 3.6 *Under the remaining hypotheses of Theorem 3.3, assume instead that*

$$\begin{cases} \cup_{\ell\le 2}M_{0,n-2\ell} \ \textit{ is connected,} & n=4\ ,\textit{or}\\ \textit{the normal bundle to } M_{0,n-2} \textit{ is trivial and } M_{0,n-4} \textit{ is connected,} & n=4\ ,\textit{or}\\ \textit{the normal bundle to } M_{0,n-2} \textit{ is trivial and } M_{0,n-4}=\emptyset\ , & n\ge 4\ ,\textit{or}\\ \cup_{\ell\le 2}M_{0,n-2\ell}=\emptyset\ , & n\ge 4\ . \end{cases} \tag{3.26}$$

Then the conclusions of Theorem 3.4 hold.

□

We note that the hypotheses of Theorem 3.6 are stable under perturbations of the metric.

3.3 Existence of the twist potential

Let us briefly turn our attention to the question of existence of the twist potential; this will be obviously the case when $H_{n-3}(M)$ is trivial. Such a hypothesis, however, excludes most situations of interest from a Lorentzian point of view if $n=4$. An alternative possibility is triviality of $H_{n-3}(\mathring\Sigma)$ — this covers, in particular, all Lorentzian space-times without black hole regions, with $M=\Sigma\times S^1$, and with trivial $H_{n-3}(\Sigma)$.

In dimension four, a further family of examples can be obtained as follows: It follows from Lemma 3.2 that, in the coordinate system of (A.1), the one forms

$$\hat\omega_A dx^A \ \text{ and } \ \omega_A dx^A$$

extend by continuity to closed one-forms $\hat\omega_0$ on ∂M, and ω_0 on $\partial\Sigma$. Clearly a necessary condition for exactness of $\hat\omega$ is exactness of $\hat\omega_0$. Under some conditions this can be shown to be sufficient:

THEOREM 3.7 *In dimension $n = 4$, suppose that X has no zeros, then the twist potentials τ and $\hat{\tau}$ exist under either of the following conditions:*

1. *There exists a function $\hat{\tau}_0$ on ∂M such that $d\hat{\tau}_0 = \hat{\omega}_0$, and there are no non-trivial L^2 sections φ of $\Lambda^1(M)$ which are solutions of the equations*

$$d\varphi = d *_g \varphi = 0 \,. \tag{3.27}$$

2. *There are no S^1 orbits with nontrivial isotropy, there exists a function τ_0 on ∂M such that $d\tau_0 = \omega_0$, and there are no non-trivial L^2 sections φ of $\Lambda^1(\mathring{\Sigma})$ which are solutions of the equations*

$$d\varphi = d *_{g_\Sigma} \varphi = 0 \,. \tag{3.28}$$

REMARK 3.8 Wang [29, Theorem 3.1] gives a condition under which the L^2–cohomology condition above will hold; in particular it follows from the work of Lee [24] that the L^2–cohomology condition will be satisfied when the Yamabe invariant of the boundary metric on ∂M or on $\partial\Sigma$ is positive.

PROOF: We first note that existence of τ and $\hat{\tau}$ are equivalent, by projecting down or lifting. In order to prove point 1 consider the equation

$$\nabla_i \nabla^i \tilde{\tau} = \frac{4}{u} \hat{\omega}_i \nabla^i u \,. \tag{3.29}$$

Lemma 3.2 and the calculations there show that $u^{-1}\hat{\omega}_i \nabla^i u = O(r^2)$, so that by point (ii) of Theorem 7.2.1 of [4], together with the Remark (i) following that theorem, there exists a function

$$\tilde{\tau} = \hat{\tau}_0 + O(r^2)$$

which solves (3.29). Equation (B.25) shows that the one-form

$$\varphi := \hat{\omega}_i dx^i - d\tilde{\tau}$$

solves (3.27). Further we have, in the coordinates of Appendix A,

$$\partial_i \rfloor \varphi = O(r) \quad \Big(\text{equivalently,} \quad |\varphi|^2_g = O(r^4)\Big) \,,$$

which implies that $\varphi \in L^2$. The vanishing of φ follows from our hypothesis of the vanishing of the first L^2–cohomology class of M, hence $\hat{\omega} = d\tilde{\tau}$. Point 2 is established in a similar way, using (B.20) instead of (B.25). □

A Extensions of conformal isometries from ∂M to M

Let Y be a conformal Killing vector field of ∂M. Suppose, first, that $(\partial M, \gamma)$ is a round sphere; as discussed at the beginning of Section 3.2, the pair (M, g) is then the hyperbolic space. Consider, next, a ∂M which is *not* a round $(n-1)$–dimensional sphere, then the Lelong-Ferrand – Obata theorem shows that we

can choose a representative $\hat{g}(0)$ of the conformal class $[\gamma]$ so that Y is a Killing vector thereof. We start with a study of the Killing equation in a neighborhood of ∂M. It is well known that there exists a defining function r such that the metric g takes the form

$$g = r^{-2}\overline{g} = r^{-2}(dr^2 + \hat{g}(r)) , \qquad \hat{g}(r)(\partial_r, \cdot) = 0 . \tag{A.1}$$

on $[0,\epsilon] \times \partial M$. Let $(x^2, ..., x^n)$ be a local coordinate system on ∂M. We will work in the coordinate system $(x1 = r, x2, ...x^n)$, and denote by r the index relative to the first coordinates. We will take upper case Latin letters for the indices relative to the remaining coordinates, and lower case Latin letters for the indices relative to any component. With that convention, the Christoffel symbols of g read

$$\Gamma^r_{rr} = -r^{-1}, \ \ \Gamma^A_{rr} = \Gamma^r_{Ar} = 0, \ \ \Gamma^r_{AB} = r^{-1}\hat{g}_{AB}(r) - \tfrac{1}{2}\hat{g}'_{AB}(r), \tag{A.2}$$

$$\Gamma^C_{rA} = -r^{-1}\delta^C_A + \tfrac{1}{2}\hat{g}^{CD}(r)\hat{g}'_{DA}(r), \ \ \Gamma^C_{AB} = \hat{\Gamma}^C_{AB}(r) . \tag{A.3}$$

Here f' denotes the derivative of a function f with respect to r. The Killing equations,

$$\nabla_i X_j + \nabla_j X_i = 0, \tag{A.4}$$

written out in detail, read

$$\partial_r X_r + r^{-1} X_r = 0, \tag{A.5}$$

$$\partial_r X_A + \partial_A X_r + 2r^{-1} X_A - \hat{g}^{CD}(r)\hat{g}'_{DA}(r) X_C = 0, \tag{A.6}$$

$$\partial_A X_B + \partial_B X_A - 2\hat{\Gamma}^C_{AB}(r) X_C + (\hat{g}'_{AB}(r) - 2r^{-1}\hat{g}_{AB}(r)) X_r = 0. \tag{A.7}$$

From (A.5) there exists a function α such that

$$X_r = \frac{\alpha}{r}, \qquad \partial_r \alpha = 0 ,$$

and, if we define $\hat{X}_A = r^2 X_A$, then (A.6) and (A.7) become

$$\partial_r \hat{X}_A + r\partial_A \alpha - \hat{g}^{CD}(r)\hat{g}'_{DA}(r)\hat{X}_C = 0, \tag{A.8}$$

$$\partial_A \hat{X}_B + \partial_B \hat{X}_A - 2\hat{\Gamma}^C_{AB}(r)\hat{X}_C + (r\hat{g}'_{AB}(r) - 2\hat{g}_{AB}(r))\alpha = 0. \tag{A.9}$$

From (A.9), $\hat{X}(0)$ is a Killing vector field of the boundary if and only if $\alpha \equiv 0 \Leftrightarrow X_r \equiv 0$. If that is the case then (A.8) and (A.9) take the form

$$\partial_r \hat{X}_A - \hat{g}^{CD}(r)\hat{g}'_{DA}(r)\hat{X}_C = 0, \tag{A.10}$$

$$\partial_A \hat{X}_B + \partial_B \hat{X}_A - 2\hat{\Gamma}^C_{AB}(r)\hat{X}_C = 0. \tag{A.11}$$

Equation (A.10) has the unique solution

$$\hat{X}_A(r) = \hat{g}_{AC}(r)\hat{X}^C(0) , \qquad \hat{X}^C(0) := \hat{g}^{CB}(0)\hat{X}_B(0). \tag{A.12}$$

We use now the Taylor development

$$\hat{g}(r) = \hat{g}(0) + O(r^p),$$

where $p = 1$ in general and $p = 2$ if g is Einstein [16]. This yields $\hat{X}(r) = \hat{X}(0)+O(r^p)$ and $\hat{\Gamma}(r) = \hat{\Gamma}(0)+O(r^p)$, thus $\hat{X}$ given by (A.12) is an approximate solution of (A.11) modulo $O(r^p)$. Finally the 1-form

$$X_\infty := 0\,dr + r^{-2}(\hat{X}_2(r)dx^2 + \ldots + \hat{X}_n(r)dx^n) \tag{A.13}$$

is an approximate solution of (A.4), with error – in the above coordinates – $O(r^{p-2})$.

We wish to show that, under reasonably mild conditions, conformal isometries of $[\gamma]$ extend to isometries of g:

Proposition A.1 *Let (M, g) be an asymptotically hyperbolic Einstein manifold and suppose that the operator*

$$\Delta_L + 2(n-1)$$

acting on symmetric two-covariant tensors has no L^2 kernel. Then every Killing vector field $\hat{X}(0)$ on ∂M extends in a unique way to a Killing vector field X on M such that (A.15) *holds.*

For $\hat{X}(0)$, a (one form associated to a) Killing vector field on ∂M, consider the boundary value problem

$$\Delta_g X_i = -\operatorname{Ric}(g)_i{}^j X_j \;, \tag{A.14}$$

$$X - X_\infty \in C^{2,\alpha}_p(M, T^*M) \;, \tag{A.15}$$

with p as defined in the paragraph before (A.13). We have

Proposition A.2 *Let (M, g) be an asymptotically hyperbolic manifold with* $\operatorname{Ric}(g) < 0$*. Then the problem* (A.14)-(A.15) *always has a unique solution.*

Proof: From Mazzeo [26] (see also [23, Lemma 7.2]) the indicial radius of the Laplace-Beltrami operator $dd^* + d^*d$ on one-forms, equal to $\nabla^*_g\nabla_g + \operatorname{Ric}(g)$ on those, is $(n-1)/2-1$. Corollary 7.4 in [23] implies then that the indicial radius of the operator

$$P = \nabla^*_g\nabla_g - \operatorname{Ric}(g) \tag{A.16}$$

on one-forms is $\sqrt{[(n-1)/2-1]^2 + 2(n-1)} = (n-1)/2+1$. An integration by parts shows that there are no C^2 compactly supported solutions of (A.14):

$$0 \le \int -\operatorname{Ric}(g)_{ij}X^iX^j = \int X_i\Delta_g X^i = -\int |\nabla X|^2 \le 0$$

(recall $\operatorname{Ric}(g) < 0$). Elliptic regularity, completeness of M together with density results (*cf., e.g.,* [6]) imply that P has trivial L^2 kernel, and [23, Theorem C] establishes that, in the notations of [23], P is an isomorphism from $C^{k,\alpha}_\delta(M; T^1) \equiv C^{k,\alpha}_\delta(M; T^*M)$ to $C^{k-2,\alpha}_\delta(M; T^*M)$ for all δ such that $|\delta - (n-1)/2| < (n-1)/2+1 \Leftrightarrow -1 < \delta < n$. (In the case of vector fields, the space $C^{k-2,\alpha}_{-1}(M; T^*M)$ corresponds to $O(r^{-2})$ behavior in the coordinates

of (A.1).) Let χ be a smooth function on M equal to 1 on $\{0 \le r \le \epsilon/3\}$, and equal to 0 for $r \ge 2\epsilon/3$; define the 1−form

$$Y = \chi X_\infty \,,$$

with X_∞ defined in (A.13). Then Y is an approximate solution to the Killing equation (A.4) modulo $O(r^{p-2})$. (We emphasise that (A.5) and (A.6) are satisfied identically near the boundary, so that the fall-off of the error term is dictated by a possible error in (A.7).) This implies $PY = O(r^{p-1}) \in C_p^{k-2,\alpha}(M;T^*M)$ (see, *e.g.*, the proof of [23, Lemma 3.7] for that last property). Thus there exists a unique solution $Z \in C_p^{k,\alpha}(M;T^*M)$ to the equation

$$PZ = -PY \;.$$

We set $X = Y + Z$; uniqueness is obvious from what has been said above. □

Proof of Proposition A.1: If we denote by $B(h) = -\mathrm{tr}\,_g \nabla h + \frac{1}{2}\nabla \mathrm{tr}\,_g(h)$, then the linearisation of the Einstein operator at the Einstein metric g is [7, Theorem 1.174]

$$D\,\mathrm{Ein}(g) = \frac{1}{2}(\Delta_L + 2(n-1)) - \mathrm{div}^*\, B.$$

Let X be the solution of (A.14)-(A.15). Then X is in the kernel of the operator P of (A.16). A two-line calculation shows that

$$B(\mathcal{L}_X g) = P(X) = 0 \;.$$

Now, if g is an Einstein metric then $\mathcal{L}_X g$ is in the kernel of $D\,\mathrm{Ein}(g)$ whatever the vector field X: if ϕ_t denotes the (perhaps local) flow of X, then

$$0 = \frac{d}{dt}(\phi_t^*(\mathrm{Ein}(g)))\Big|_{t=0} = \frac{d}{dt}(\mathrm{Ein}(\phi_t^* g))\Big|_{t=0} = D\,\mathrm{Ein}(g)\mathcal{L}_X g \;.$$

It thus follows that

$$(\Delta_L + 2(n-1))\mathcal{L}_X g = 0 \;.$$

Now, Theorem C and Proposition D of [23] show that the operator $\Delta_L + 2(n-1)$ is an isomorphism from $C_\delta^{k,\alpha}(M,S_2)$ to $C_\delta^{k-2,\alpha}(M,S_2)$ for all δ such that

$$|\delta - (n-1)/2| < (n-1)/2 \quad \Longleftrightarrow \quad 0 < \delta < n-1 \;.$$

Here we use the symbol S_2 to denote the bundle of symmetric two-covariant tensors; in the notation of [23] the space $C_0^{k,\alpha}(M,S_2)$ corresponds to $O(r^{-2})$ behavior in the coordinates of (A.1). From what has been said we have $\mathcal{L}_X g = O(r^{p-2})$ in local coordinates near the boundary, which can be written as $\mathcal{L}_X g \in C_p^{k,\alpha}(M,S_2)$. Since $p > 0$ the isomorphism property gives

$$\mathcal{L}_X g = 0.$$

□

REMARK A.3 As X is a Killing vector field, $\hat{X}$ satisfies (A.10) and (A.11), in particular the field of covectors $\hat{X}_A(r_0) = \hat{g}_{AC}(r_0)\hat{g}^{CB}(0)\hat{X}_B(0)$ satisfies the Killing equations on the hypersurface $\{r = r_0\}$. This is equivalent to the statement that X is tangent to the level sets of r with $\hat{X}^A(r) = \hat{X}^A(0)$. Further, in the coordinate system of (A.1),

$$\xi := g(X, \cdot) = r^{-2}(\hat{X}_2(r)dx^2 + \ldots + \hat{X}_n(r)dx^n) \; . \tag{A.17}$$

B The norm and twist equations

Let (M, g) be an n-dimensional Riemannian or Lorentzian space-time with a Killing vector field X,

$$\nabla_i X_j + \nabla_j X_i = 0 \; . \tag{B.1}$$

It is well known that (B.1) implies the equation

$$\nabla_i \nabla_j X^k = R_{\ell ij}{}^k X^\ell \; , \tag{B.2}$$

in particular

$$\nabla^j \nabla_j X^k = -\operatorname{Ric}^k{}_j X^j \; . \tag{B.3}$$

Let us, locally, write the metric in the form (3.1):

$$g = \eta u^2 (d\phi + \theta)^2 + g_\Sigma \; , \quad \theta(\partial_\phi) = g_\Sigma(\partial_\phi, \cdot) = 0 \; , \quad X = \partial_\phi \; , \tag{B.4}$$

where g_Σ is the metric induced by g on the distribution $X^\perp \subset TM$, and $\eta = \pm 1$ according to whether X is spacelike ($\eta = 1$) or timelike ($\eta = -1$). The metric g_Σ is the natural metric on the orbit space Σ [19]. One can also think of Σ as of any hypersurface transverse to X, regardless of the structure of the flow of X; one should then, however, not confuse g_Σ with the metric induced by g on Σ. We will be interested in the equations on Σ; an efficient way of obtaining those is provided by the projection formalism of Geroch [19]. We will be working away from the set of zeros of $g(X, X)$. Let

$$P : TM \to TM$$

denote the orthogonal projection on $X^\perp$, we will also use the symbol P to denote the obvious extension of P to other tensor bundles. We note that

$$u = \sqrt{\eta g(X, X)} \; , \tag{B.5}$$

(which can be used as the definition of u regardless of the decomposition (B.4)) and we set

$$n := \frac{X}{u} \; . \tag{B.6}$$

We then have

$$P(Y) = Y - \eta g(Y, n) n = (\delta^i_j - \eta n^i n_j) Y^j \partial_i \; .$$

If Y and Z are tangent to Σ, and if $\hat{Y}$ and $\hat{Z}$ are X-orthogonal, X-invariant lifts of Y and Z to M, then the covariant derivative defined as

$$D_Y Z := P(\nabla_{\hat{Y}} \hat{Z})$$

is the Levi-Civita covariant derivative of g_Σ (see [19]). Let

$$\hat{\lambda} := P(u \nabla X) \ ,$$

so that

$$\hat{\lambda}_{ij} = u \nabla_i X_j + X_i u_j - X_j u_i \ , \tag{B.7}$$

where we have written u_j for $\nabla_j u$. The tensor field $\hat{\lambda}$ is well defined and smooth away from the set of zeros of u (at which u might fail to be differentiable). One has $X^i \nabla_i u = 0$ by (B.1), and one easily checks that $\hat{\lambda}$ is an anti-symmetric X-invariant tensor field on M which annihilates X, and thus defines a two-form λ on Σ in the usual way. A convenient way of calculating λ in practice is to introduce

$$\beta := u^{-2} \xi \ . \tag{B.8}$$

With a little work one finds

$$\hat{\lambda} = u^3 d\beta \ , \tag{B.9}$$

which clearly leads to

$$d(u^{-3} \hat{\lambda}) = 0 \ . \tag{B.10}$$

It can be seen that X is (locally) hypersurface orthogonal if and only if $\hat{\lambda}$ vanishes. Indeed, (B.4) shows that the distribution $X^\perp$ is (locally) integrable if and only if

$$d\phi + \theta = \frac{\eta}{u^2} g(X, \cdot)$$

is closed; that last condition is precisely the equation $\hat{\lambda} = 0$.

Let us derive the equations satisfied by $\hat{\lambda}$ and λ. Using (B.3) we have

$$\nabla_k \hat{\lambda}_{ij} = u_k \nabla_i X_j + u R_{skij} X^s + \nabla_k X_i u_j - \nabla_k X_j u_i + X_i \nabla_k u_j - X_j \nabla_k u_i \ . \tag{B.11}$$

Applying a projection to both sides of (B.11) one finds

$$D_k \lambda_{ij} = \frac{1}{u} u_k \lambda_{ij} + u P(R_{skij} X^s) + \frac{1}{u} (\lambda_{ki} u_j - \lambda_{kj} u_i) \ . \tag{B.12}$$

Projections commute with anti-symmetrisations, so that the first Bianchi identity implies

$$D_{[k} \lambda_{ij]} = \frac{3}{u} u_{[k} \lambda_{ij]} \ ,$$

where square brackets denote complete anti-symmetrisation. Equivalently,

$$d(u^{-3} \lambda) = 0 \ , \tag{B.13}$$

where d is taken on $\mathring{\Sigma}$. This does imply (B.10) by pull-back with π_Σ, but the implication the other way round does not seem to be evident.

We want to calculate the divergence of λ. In order to do that we need to work out the $P(R_{skij}X^s)$ term appearing in (B.12); using the fact that n is proportional to X we find

$$\begin{aligned} P(R_{skij}X^s) &= P((\delta^\ell_j - n_j n^\ell)R_{ski\ell}X^s) \\ &= P((R_{skij} - n_j n^\ell R_{ski\ell})X^s) \\ &= P((\delta^m_i - n_i n^m)(R_{skmj} - n_j n^\ell R_{skm\ell})X^s) \\ &= (R_{skij} - n_i n^m R_{skmj} - n_j n^\ell R_{ski\ell})X^s \ , \end{aligned}$$

where the last equality arises from the fact that all projections have already been carried out; no projection is needed in the k index since n is proportional to X. Upon a contraction over k and i in (B.12) the $P(R_{skij}X^s)$ term will thus give a contribution

$$(-R_{sj} - 0 + n_j n^\ell R_{s\ell})X^s = 0$$

if g is Einstein. It follows that, for Einstein metrics g,

$$D^i\lambda_{ij} = \frac{1}{u}(u^i\lambda_{ij} + 0 + \underbrace{\lambda_i{}^i}_{0} u_j - \lambda_{ij}u^i) = 0 \ . \tag{B.14}$$

Equivalently,

$$d(*_{g_\Sigma}\lambda) = 0 \ , \tag{B.15}$$

with d again taken on $\mathring{\Sigma}$. We define the *twist* $n-3$ *form* ω *on* $\mathring{\Sigma}$ by the equation

$$\omega := *_{g_\Sigma}\lambda \ . \tag{B.16}$$

Equation (B.15) shows that ω is closed, while (B.13) is equivalent to

$$d^*(u^{-3}\omega) = 0 \ , \tag{B.17}$$

Let $\hat\omega$ denote the lift of ω to M,

$$\hat\omega = \pi^*_\Sigma\omega \ ,$$

where π_Σ is the projection from M to Σ. Choosing the orientation of Σ appropriately one finds

$$\hat\omega_{\alpha_1\ldots\alpha_{n-3}} = \epsilon_{\alpha_1\ldots\alpha_{n-3}\alpha\beta\gamma}X^\alpha\nabla^\beta X^\gamma \quad \Longleftrightarrow \quad \hat\omega = *_g(\xi\wedge d\xi) \ , \tag{B.18}$$

where

$$\xi = g(X,\cdot) \ ,$$

and where ϵ is the volume form on M. Since exterior differentiation commutes with pull-back we have

$$d\hat\omega = d(\pi^*_\Sigma\omega) = \pi^*_\Sigma(d\omega) = 0 \ . \tag{B.19}$$

Summarising, on Σ we have

$$d\omega = d(u^{-3} *_{g_\Sigma} \omega) = 0 \ , \tag{B.20}$$

while on M it holds that

$$d\hat{\omega} = d(u^{-3}\hat{\lambda}) = 0 \; . \tag{B.21}$$

It is worthwhile mentioning that so far all the equations were manifestly signature-independent.

We note the following equations for u:

$$\begin{aligned}
\nabla_i u &= \frac{\eta}{u} X^j \nabla_i X_j \; , \\
\nabla^i \nabla_i u &= \frac{\eta}{u}(-\eta g(\nabla u, \nabla u) + \nabla^i X^j \nabla_i X_j - \mathrm{Ric}_{ij}\, X^i X^j) \\
&= \frac{1}{u}(g(\nabla u, \nabla u) + \eta \frac{\hat{\lambda}^{ij}\hat{\lambda}_{ij}}{u^2} - \eta\, \mathrm{Ric}_{ij}\, X^i X^j) \; .
\end{aligned} \tag{B.22}$$

The reader is warned that the $\hat{\lambda}^{ij}\hat{\lambda}_{ij}$ term above can sometimes be negative when g is Lorentzian and X is spacelike; similarly $g(\nabla u, \nabla u)$ can sometimes be negative for Lorentzian metrics.

We refer to [14] for explicit formulae for the curvature tensor of g_Σ.

For completeness let us recall how this formalism works in dimension four: one then sets

$$\omega_i \;\; := \;\; \epsilon_{ijk\ell} X^j \nabla^k X^\ell \; . \tag{B.23}$$

Here, as before, ϵ_{ijkl} is the volume form,

$$\epsilon_{ijkl} = 0, \pm\sqrt{|\det g_{mn}|} \; ,$$

with ϵ_{ijkl} totally antisymmetric, the sign being $+$ for positive permutations of 1234. The form ω from (B.23) is actually the form $\hat{\omega}$ from (B.18), but we shall not make a distinction between ω and $\hat{\omega}$ anymore. One has $X^i\omega_i = 0$ by antisymmetry of $\epsilon_{ijk\ell}$. Working away from the set of zeros of u, with a little work one finds

$$\nabla_i X_j = \frac{2}{u} X_{[j} \nabla_{i]} u + \frac{\sigma\eta}{2u^2} \epsilon_{ijk\ell}\omega^k X^\ell \; , \tag{B.24}$$

where $\sigma = +1$ in the Riemannian case, and $\sigma = -1$ in the Lorentzian one. The simplest way of performing the algebra involved in this equation, as well as in (B.25) below, is to consider a frame in which $X = ue^1$, with ω proportional to e^2. Comparing with (B.7), one recognises the last term above as λ_{ij}. The divergence of $\hat{\omega}$ can also be computed directly as follows:

$$\begin{aligned}
\nabla^i \hat{\omega}_i &= \epsilon_{ijk\ell}(\nabla^i X^j \nabla^k X^\ell + X^j R_m{}^{ik\ell} X^m) \\
&= \epsilon_{ijk\ell} \nabla^i X^j \nabla^k X^\ell \\
&= \frac{4\hat{\omega}_i}{u} \nabla^i u \; .
\end{aligned} \tag{B.25}$$

Equivalently,

$$\nabla^i (u^{-4}\hat{\omega}_i) = 0 \; .$$

Equation (B.19) can be rewritten as

$$\nabla_i \hat{\omega}_j - \nabla_j \hat{\omega}_i = D_i \omega_j - D_j \omega_i = 0 \; . \tag{B.26}$$

It follows that

$$\begin{aligned}\nabla^k\nabla_k\hat\omega_i &= \nabla^k\nabla_i\hat\omega_k = \nabla_i\nabla^k\hat\omega_k + \mathrm{Ric}_{ij}\,\hat\omega^j\\ &= 4\nabla_i(\frac{\hat\omega_j\nabla^j u}{u}) + \mathrm{Ric}_{ij}\,\hat\omega^j\ .\end{aligned}\tag{B.27}$$

In the four-dimensional case the last line of (B.22) can be rewritten as

$$\nabla^i\nabla_i u = \frac{1}{u}(g(\nabla u,\nabla u) + \frac{1}{2u^2}\sigma g(\hat\omega,\hat\omega) - \eta\,\mathrm{Ric}_{ij}\,X^iX^j)\ .\tag{B.28}$$

C The structure of the orbit space near fixed points

In order to analyse the contribution to (3.16) arising from the integral over $\partial\Sigma_{\mathrm{sing},0}(\rho)$, we need to recall some results about the structure of $\partial\Sigma_{\mathrm{sing},0}$. Since ∇_iX_j is antisymmetric, for every $p\in M$ there exists an ON basis of T_pM in which ∇_iX_j is block-diagonal, with ℓ non-zero anti-symmetric two-by-two blocks eventually followed by a block of zeros; such a basis will be referred to as *a basis adapted to* ∇X. It follows that the dimension of the set

$$\mathrm{Ker}_p\nabla X := \{Y\in T_pM : \nabla_Y X = 0\}\tag{C.1}$$

is necessarily a number of the form $n-2\ell$ for some $0\le\ell\le n/2$. For such ℓ's we define

$$M_{0,n-2\ell} := \{p\in M : X(p)=0\ ,\ \dim(\mathrm{Ker}_p\nabla X) = n-2\ell\}\ .\tag{C.2}$$

For $p\in M$ let $\mathrm{Iso}(p)$ denote the isotropy group of p. We set

$$M_{\mathrm{iso}} := \{p\in M :\ X(p)\neq 0\ ,\ \mathrm{Iso}(p)\neq \mathrm{Id}\}\ .\tag{C.3}$$

For $p\in M_{\mathrm{iso}}$ let $\tau_p\in\{2\pi/n\}_{n\in\mathbb{N}^*}$ denote the period of the orbit of X through p, set

$$\begin{aligned}\mathrm{Inv}_p &:= \{Y\in T_pM : (\phi_{\tau_p})_*Y = Y\}\ ,\\ M_{\mathrm{iso},\ell} &:= \{p\in M_{\mathrm{iso}} :\ \dim(\mathrm{Inv}_p) = \ell\}\ .\end{aligned}\tag{C.4}$$

The following is certainly well known; we give the proof for completeness, because some elements of the argument will be needed in our further analysis:

PROPOSITION C.1 *1. The $M_{\mathrm{iso},\ell}$'s are smooth, totally geodesic ℓ–dimensional submanifolds of M.*

2. The $M_{0,n-2\ell}$'s are smooth, closed, totally geodesic $(n-2\ell)$–dimensional submanifolds of M. In particular

$$M_{0,n-2i}\cap M_{0,n-2j} = \emptyset\quad \textit{for}\quad i\neq j\ .$$

PROOF: 1. Let $p \in M_{\mathrm{iso},\ell}$ and let $\gamma : [0, s_p) \to M$ be a maximally extended distance-parameterised geodesic such that $\gamma(0) = p$ and $\dot\gamma(0) = Y$ for some $Y \in T_pM$. If $Y \in \mathrm{Inv}_p$, then $\phi_{\tau_p} \circ \gamma : [0, s_p) \to M$ is again a maximally extended geodesic through p with tangent vector Y, which implies $\phi_{\tau_p} \circ \gamma(s) = \gamma(s)$ for all $s \in [0, s_p)$. It follows that the group orbit through $\gamma(s)$ has period τ_p for s small enough. Clearly if $Y \notin \mathrm{Inv}_p$ then we will have $\phi_{\tau_p} \circ \gamma(s) \neq \gamma(s)$ again for s small enough, and the result follows.

2. For $s \in \mathbb{R}$ let ϕ_s denote the action of S^1 on M, with s normalised so that 2π is the smallest strictly positive number s_* for which ϕ_{s_*} is the identity on M. At points at which X vanishes we have, for any vector field Y,

$$\mathscr{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X = -\nabla_Y X \ ,$$

so that

$$\frac{d(\phi_{s*}Y)}{ds} = 0 \quad \text{for} \quad Y \in \mathrm{Ker}_p \nabla X \ . \tag{C.5}$$

Consider any maximally extended affinely parameterised geodesic $\gamma : I \to M$ with $\gamma(0) = p$, and with the tangent $\dot\gamma(0) \in \mathrm{Ker}_p\nabla X$. Then $\phi_s(\gamma)$ is again a maximally extended affinely parameterised geodesic through p. Further,

$$\frac{d(\phi_{s*}\dot\gamma(0))}{ds} = 0 \tag{C.6}$$

by (C.5), which shows that

$$\forall s \quad \frac{d(\phi_s \circ \gamma)(t)}{dt}\Big|_{t=0} = \dot\gamma(0) \ .$$

This implies of course that $\phi_s(\gamma(t)) = \gamma(t)$ for all $t \in M$ and $s \in \mathbb{R}$, so that all points on γ are fixed points of ϕ_s. Hence

$$\exp_p(\mathrm{Ker}_p \nabla X) \subset \cup_\ell M_{0,n-2\ell} \ .$$

If we move away from p in a direction which is not in $\mathrm{Ker}_p\nabla X$ then X immediately becomes non-zero, which shows that there exists a neighborhood of p such that $\exp_p(\mathrm{Ker}_p\nabla X)$ coincides with $M_{0,n-2\ell}$ there, and the fact that $M_{0,n-2\ell}$ is a smooth embedded totally geodesic submanifold follows.

To prove closedness of $M_{0,n-2\ell}$ consider normal coordinates centred at p. After performing a rotation if necessary we may suppose that the basis $\{\partial_i\}$ is adapted to ∇X at p, so that there exist real numbers $\kappa_i = \kappa_i(p)$ such that at p we have

$$\frac{1}{2}\nabla_i X_j \, dx^i \wedge dx^j = \sum_{i=1}^{\ell} \kappa_i \, dx^{2i-1} \wedge dx^{2i} \ . \tag{C.7}$$

Closedness of $M_{0,n-2\ell}$ is clearly equivalent to the statement that the $|\kappa_i|$'s are uniformly bounded away from zero on each of the $M_{0,n-2\ell}$'s. It is shown below that the κ_i's are integers, and continuity of the map $M_{0,n-2\ell} \ni p \to \{\kappa_i(p)\} \in \mathbb{R}^\ell$ proves the result. $\square$

In order to continue our analysis of the geometry near fixed points let $p \in M_{0,n-2\ell}$, with $M_{0,n-2\ell}$ as in (C.2), let x^a denote any local coordinates on $M_{0,n-2\ell}$

on a coordinate $M_{0,n-2\ell}$–neighborhood $\mathscr{O}_p \subset M_{0,n-2\ell}$ of p, and for $q \in \mathscr{O}_p$ let x^A denote geodesic coordinates on $\exp_q\{(T_qM_{0,n-2\ell})^\perp\}$. Passing to a subset of $\mathscr{O}_p$ if necessary one obtains thus a coordinate system $(x^i) = (x^A, x^a)$, with $A = 1, \ldots, 2\ell$, on an M–neighborhood $\mathscr{U}_p \subset M$ of p diffeomorphic to $\mathscr{O}_p \times B_{2\ell}(r)$, where $B_{2\ell}(r)$ is a ball of radius r centred at the origin in $\mathbb{R}^{2\ell}$. Since $M_{0,n-2\ell}$ is compact, it can be covered by a finite number of such coordinate systems. This leads to the following local form of the metric

$$g = \sum_{i=1}^{2\ell}(dx^i)^2 + h + \sum_{A,a} O(\rho)dx^A dx^a + \sum_{A,B} O(\rho^2)dx^A dx^B + \sum_{a,b} O(\rho^2)dx^a dx^b \ , \tag{C.8}$$

with h — the metric induced by g on $M_{0,n-2\ell}$, where ρ denote the geodesic distance to $M_{0,n-2\ell}$. The $O(\rho^2)$ character of the $dx^A dx^B$ error terms is standard; the $O(\rho^2)$ character of the $dx^a dx^b$ error terms follows from the totally geodesic character of $M_{0,n-2\ell}$. We shall need an anti-symmetry property of the derivatives of the g_{aA}'s, which we now derive: by construction, the coordinate rays $s \to sx^A$ are affinely parameterised geodesics. This gives

$$0 = \frac{d^2x^a}{ds^2} + \Gamma^a_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = \Gamma^a_{BC}\frac{dx^A}{ds}\frac{dx^B}{ds} \ .$$

Since the vector dx^A/ds can be arbitrarily chosen at $s = 0$ this implies

$$0 = \Gamma^a_{BC}|_{x^A=0} \quad \Longleftrightarrow \quad (g_{aA,B} + g_{aB,A})|_{\{x^C=0\}} = 0 \ , \tag{C.9}$$

where a comma denotes a partial derivative. (Similar arguments may of course be used to justify the $O(\rho^2)$ character of the remaining error terms in (C.8).)

Exponentiating (C.7) shows that on each space $(T_qM_{0,n-2\ell})^\perp$ the one-parameter group of diffeomorphisms ϕ_s generated by X acts as a rotation of angle $\kappa_i s$ of the planes $\mathrm{Vect}\{\partial_{2i-1}, \partial_{2i}\}$, $1 \le i \le \ell$. The definition of geodesic coordinates implies that on $\mathscr{U}_p$ the Killing vector X equals

$$X = \sum_{i=1}^{\ell} \kappa_i(x^{2i-1}\partial_{2i} - x^{2i}\partial_{2i-1}) \ . \tag{C.10}$$

This equation is exact; there are no error terms, as opposed to *e.g.* (C.8). Since $\phi_{2\pi}$ is the identity we have $\kappa_i \in \mathbb{Z}^*$, and since almost all orbits have period 2π it follows that at least one $|\kappa_i|$ — say $|\kappa_1|$ — equals one. For $\ell \ge 2$ by renaming and multiplication by -1 of the coordinates one can arrange to have

$$1 = |\kappa_1| \le \kappa_i \le \kappa_{i+1} \le \kappa_\ell \ , \quad 2 \le i \le \ell - 1 \ , \tag{C.11}$$

and we will always assume that (C.11) holds. We have assumed that an orientation of $(T_qM_{0,n-2\ell})^\perp$ has been chosen, and the sign in κ_1 is chosen so that the coordinates of (C.10) have the correct orientation. Continuity shows that the κ_i's are constant over each connected component of $M_{0,n-2\ell}$.

We shall denote by ρ_i and φ_i the polar coordinates in the (x^{2i-1}, x^{2i}) planes:

$$x^{2i-1} = \rho_i \cos\varphi_i \ , \qquad x^{2i} = \rho_i \sin\varphi_i \ , \tag{C.12}$$

so that

$$X = \sum_{i=1}^{\ell} \kappa_i \frac{\partial}{\partial \varphi_i} \ . \tag{C.13}$$

If all the κ_i's are ones, then all orbits in $\mathscr{U}_p$ have period 2π, in which case

$$M_{\mathrm{iso}} \cap \mathscr{U}_p = \pi_\Sigma^{-1}(\Sigma_{\mathrm{sing,iso}}) \cap \mathscr{U}_p = \emptyset \ .$$

Otherwise $\ell \geq 2$ and there exists a smallest i_2 such that $\kappa_i \geq 2$ for $i \geq i_2$. If q is such that $\rho_i(q) = 0$ for $1 \leq i < j$, and $\rho_j(q) > 0$, then the orbit of X through p has period $2\pi/\kappa_j$. It follows that an orbit through $q \in \mathscr{U}_p$ has trivial isotropy if and only if

$$\sum_{i=1}^{i_2-1} \rho_i(q) \neq 0 \ .$$

We have shown:

Proposition C.2 *We have*

$$M_{0,n-2\ell} \cap \overline{M_{\mathrm{iso}}} \neq \emptyset \quad \Longleftrightarrow \quad \exists\, i \text{ such that } \kappa_i \geq 2 \ ,$$

in particular

$$M_{0,n-2} \cap \overline{M_{\mathrm{iso}}} = \emptyset \ .$$

To proceed further, we need to understand the structure of Σ near $\pi_\Sigma M_{0,n-2\ell}$. We first use the polar coordinates (C.12), and then introduce new angular variables φ, ψ_i, parameterising $\underbrace{S^1 \times \cdots \times S^1}_{n \text{ factors}}$, defined as

$$\varphi := \varphi_1 \ , \qquad \psi_i := \varphi_i - \kappa_1 \kappa_i \varphi_1 \ , \ i = 2, \ldots, n \ , \tag{C.14}$$

so that, using (C.13),

$$X(\varphi) = 1 \ , \qquad X(\psi_i) = X(\rho_i) = 0 \ .$$

It follows that $X = \partial_\varphi$, and that $(\rho_1, (\rho_i, \psi_i)_{i \geq 2})$, can be used as local coordinates on $\mathring{\Sigma}$. There is a usual "polar coordinates singularity" at the sets $\{\rho_i = 0, u \neq 0\}$ for $i \geq 2$. As already pointed out, for i's such that $\kappa_{i+1} > \kappa_i$ the periodicity of the φ_i's jumps down from $2\pi/\kappa_i$ to $2\pi/\kappa_{i+1}$ at the sets $\{\rho_1 = \ldots = \rho_i = 0, \ u \neq 0\}$. This leads to an identical jump of the periodicity of the ψ_i's, leading to orbifold singularities of increasing complexity at each of those sets. In conclusion, within the domain of the coordinate system $(\rho_1, (\rho_i, \psi_i)_{i \geq 2})$ the differentiable part $\mathring{\Sigma}$ of Σ takes the form

$$\{u > 0\}$$

if all the κ_i's are ones, and

$$\{u > 0\} \setminus \{\rho_1 = \ldots = \rho_{i_2} = 0\}$$

otherwise. Further, $(\rho_1,(\rho_i,\psi_i)_{i\geq 2})$ provide a well behaved coordinate system of polar type on $\mathring{\Sigma}$ in a neighborhood of $\pi_\Sigma M_{0,n-2\ell}$.

Equations (C.8) and (C.10) imply

$$\xi := g(X,\cdot) = \sum_{i=1}^{\ell} \kappa_i (x^{2i-1}dx^{2i} - x^{2i}dx^{2i-1}) + \nu_a dx^a + \sum_i O(\rho^3)dx^i \ , \quad \text{(C.15)}$$

where

$$\nu_a := g_{aA,B}|_{\{x^C=0\}} X^A x^B \ . \quad \text{(C.16)}$$

At this stage it is adequate to enquire about the geometric character of the objects defined so far. Note that the locally defined coordinates x^A appearing in (C.8) are only determined modulo x^a–dependent rotations:

$$x^A \to \bar{x}^A := \omega^A{}_B(x^a)x^B \ , \quad \text{(C.17)}$$

where, at each x^a, $\omega^A{}_B$ is an 2ℓ by 2ℓ orthogonal matrix that preserves all the spaces $\mathrm{Vect}\{\partial_{2i-1},\partial_{2i}\}$. Suppose, thus, that two coordinate systems $(\bar{x}^A,\bar{x}^a)$ and (x^A,x^a) are given, with $\bar{x}^a = x^a$, and with $\bar{x}^A$ related to x^A via (C.17). It is convenient to put bars on g_{AB}, ν_a, *etc*, to denote those objects in the barred coordinate system. One easily finds the following transformation law under (C.17):

$$\frac{\partial \bar{g}_{aA}}{\partial \bar{x}^B}|_{\{\bar{x}^C=0\}} \to \frac{\partial g_{aA}}{\partial x^B}|_{\{x^C=0\}} = \sum_D \omega^D{}_A \left(\frac{\partial \bar{g}_{aD}}{\partial \bar{x}^E}|_{\{\bar{x}^C=0\}} \omega^E{}_B + \omega^D{}_{B,a} \right) . \quad \text{(C.18)}$$

It follows that

$$\bar{\nu}_a \to \nu_a = \bar{\nu}_a + \sum_D \omega^D{}_{B,a}\omega^D{}_A X^A x^B \ . \quad \text{(C.19)}$$

Now, ω is a block-diagonal matrix consisting of two-by-two blocks, each of them of the form

$$\begin{bmatrix} \cos(\theta_i(x^a)) & -\sin(\theta_i(x^a)) \\ \sin(\theta_i(x^a)) & \cos(\theta_i(x^a)) \end{bmatrix} .$$

Inserting this into (C.19) one obtains

$$\bar{\nu}_a \to \nu_a = \bar{\nu}_a + \sum_{i=1}^{\ell} \kappa_i \left((x^{2i-1})^2 + (x^{2i})^2 \right) \frac{\partial \theta_i}{\partial x^a} \ . \quad \text{(C.20)}$$

So far we have assumed that $\bar{x}^a = x^a$; this last restriction is removed in a straightforward way, leading to a tensorial transformation law of the right-hand-side of (C.20) under the transformation

$$(\bar{x}^A,\bar{x}^a) \to (\tilde{x}^A = \bar{x}^A, \tilde{x}^a = \phi^a(\bar{x}^b)) \ .$$

It should be emphasised that in general we will not be able to achieve $\omega = \mathrm{id}$ when going from one coordinate patch x^a to another on $M_{0,n-2\ell}$. This implies that $\nu_a dx^a$ does *not* transform as a one-form when passing from one x^a–coordinates patch on $M_{0,n-2\ell}$ to another, except in the case in which the

x^A's can be globally "synchronised" over $M_{0,n-2\ell}$ — this occurs if and only if each of the bundles $\mathrm{Vect}\{\partial_{2i-1}, \partial_{2i}\}$ is trivial.

In order to evaluate the $\partial\Sigma_{\mathrm{sing,iso}}(\rho)$ integral in (3.16) we need to calculate $u^{-3} *_{g_\Sigma} \omega = u^{-3}\lambda$, with λ being defined as the Σ–equivalent of the two–form $\hat\lambda$ of (B.7). The simplest way of doing this proceeds via the calculation of the form β of (B.8), *cf.* (B.9). Consider, first, the case $\ell = 1$, set

$$\gamma_a := g_{a1,2}|_{\{x^C=0\}} \; ; \tag{C.21}$$

the anti-symmetry property (C.9) gives

$$\nu_a = \gamma_a \rho_1^2 \; ,$$

hence

$$\xi = \kappa_1 \rho_1^2 d\varphi + \gamma_a \rho_1^2 dx^a + \sum_i O(\rho^3) dx^i \; , \tag{C.22}$$

so that

$$\beta := u^{-2}\xi = \kappa_1 d\varphi + \gamma_a dx^a + O(\rho) d\rho_1 + \sum_a O(\rho) dx^a \; . \tag{C.23}$$

Equation (C.20) shows that $\gamma_a dx^a$ is a connection form on the $U(1)$–principal bundle of unit vectors normal to $M_{0,n-2}$:

$$\bar\gamma_a \to \gamma_a = \bar\gamma_a + \frac{\partial \theta_1}{\partial x^a} \; . \tag{C.24}$$

In particular the curvature two-form

$$F = d\gamma$$

is a well-defined two-form on $M_{0,n-2}$.

Let us return to (C.15)-(C.16) for general $\ell \geq 2$; Equation (C.14) gives

$$\begin{aligned} \xi &= \kappa_1 \rho_1^2 d\varphi + \sum_{i=2}^{\ell} \kappa_i \rho_i^2 (d\psi_i + \kappa_1 \kappa_i d\varphi) + \nu_a dx^a + \sum_i O(\rho^3) dx^i \\ &= u^2 \Big(\kappa_1 d\varphi + \sum_{i=2}^{\ell} \kappa_i u^{-2} \rho_i^2 d\psi_i + u^{-2} \nu_a dx^a \\ &\qquad + \sum_{i\geq 1} O(\rho) d\rho_i + \sum_{i \geq 2} O(\rho^2) d\psi_i + \sum_a O(\rho) dx^a \Big) \; , \end{aligned} \tag{C.25}$$

with

$$u = \sqrt{\sum_{i=1}^{\ell} \kappa_i^2 \rho_i^2 + O(\rho^3)} \; . \tag{C.26}$$

Equations (B.8)-(B.9) together with (B.16) and (C.25) immediately lead to

$$\begin{aligned} u^{-3} *_{g_\Sigma} \omega &= u^{-3}\lambda \\ &= \sum_{i=2}^{\ell} d\left(\frac{\kappa_i \rho_i^2}{u^2} d\psi_i\right) + d(u^{-2}\nu_a dx^a) \\ &\quad + \sum_{i,j\ge 1} O(1) d\rho_j \wedge d\rho_i + \sum_{i\ge 2, j\ge 1} O(\rho) d\rho^j \wedge d\psi_i + \sum_{i,j\ge 2\ge 1} O(\rho^2) d\psi_j \wedge d\psi_i \\ &\quad + \sum_{a,i} O(1) d\rho_i \wedge dx^a + \sum_{a,j} O(\rho) d\psi_j \wedge dx^a + \sum_{a,b} O(\rho) dx^a \wedge dx^b \ , \end{aligned} \tag{C.27}$$

which is used in the proof of Theorem 3.4.

We note the following necessary condition for staticity:

PROPOSITION C.3 *If (M,g) is static, then $M_{0,n-2\ell} = \emptyset$ for $\ell > 1$.*

PROOF: Calculating directly from (C.15) we find

$$d\xi = -2\sum_{i=1}^{\ell} \kappa_i \, dx^{2i-1} \wedge dx^{2i} + d\nu_a \wedge dx^a + \sum_i O(\rho^2) dx^i \ ,$$

so that

$$\begin{aligned} d\xi \wedge \xi &= -2 \sum_{i\ne j=1}^{\ell} \kappa_i \kappa_j \, dx^{2j-1} \wedge dx^{2j} \wedge (x^{2i-1} dx^{2i} - x^{2i} dx^{2i-1}) \\ &\quad + \sum_{A,B,a} O(\rho^2) dx^A \wedge dx^B \wedge dx^a + \sum_{i,j,k} O(\rho^3) dx^i \wedge dx^j \wedge dx^k \ , \end{aligned} \tag{C.28}$$

which clearly does never vanish when $\ell \ge 2$ on a sufficiently small neighborhood of $M_{0,n-2\ell}$. □

D A family of non-degenerate black hole solutions

D.1 An injectivity theorem

We start by proving point (a) of Theorem 2.2:

THEOREM D.1 *Let $K_{\max}(x)$ and $K_{\min}(x)$ denote the largest and the smallest sectional curvature of g at x. If for all $x \in M$ it holds that either $K_{max}(x) \le 0$ or $K_{min}(x) \ge -2(n-1)/n$, then the operator $\Delta_L + 2(n-1)$ has trivial L^2 kernel.*

PROOF: We use the notations of Lee [23], except that we work in dimension n, not $n+1$. For all $x \in M$, let

$$a(x) = \sup\{(\mathring{R}m_x h_x, h_x)/|h_x|^2 \ , \ h \in S^2_0\}.$$

From [7, Lemma 12.71] we have that

$$a(x) \le \min\{(n-2)K_{max}(x) + n - 1, -(n-1) - nK_{min}(x)\} ,$$

showing that under the current hypotheses we have $n-1-a(x) \ge 0$. The proof in [23, p. 67] establishes then that the operator $\Delta_L + 2(n-1)$ has trivial kernel.
□

D.2 Sectional curvatures of generalised Kottler metrics

We consider a generalised Kottler metric,

$$g = \frac{1}{V(r)}dr^2 + V(r)d\theta^2 + r^2\hat{g} , \tag{D.1}$$

where $\hat{g} = \hat{g}_{AB}dx^A dx^B$ does not depend on r and θ. The non-trivial components of the Riemann tensor are

$$\begin{aligned} R_{r\theta r\theta} &= -\frac{1}{2}V'' \\ R_{rArB} &= -\frac{rV'}{2V}\hat{g}_{AB}, \\ R_{\theta A\theta B} &= -\frac{rV'V}{2}\hat{g}_{AB}, \\ R_{ABCD} &= r^2\hat{R}_{ABCD} - r^2V(\hat{g}_{AC}\hat{g}_{BD} - \hat{g}_{AD}\hat{g}_{BC}). \end{aligned}$$

In particular if $V(r) = c + r^2 - 2mr^{-(n-3)}$, we obtain

$$\begin{aligned} R_{r\theta r\theta} &= -1 + (n-3)(n-2)mr^{-(n-1)}, \\ R_{rArB} &= -[(r^2 + m(n-3)r^{-(n-3)})/V]\hat{g}_{AB}, \\ R_{\theta A\theta B} &= -(r^2 + m(n-3)r^{-(n-3)})V\hat{g}_{AB}, \\ R_{ABCD} &= r^2\hat{R}_{ABCD} - r^2V(\hat{g}_{AC}\hat{g}_{BD} - \hat{g}_{AD}\hat{g}_{BC}). \end{aligned}$$

Let $U = (U^r, U^\theta, U^A) = (U^r, U^\theta, \hat{U})$ and $W = (W^r, W^\theta, W^A) = (W^r, W^\theta, \hat{W})$ be two orthogonal vectors with norm 1, the sectional curvature of span(U, W) is

$$\begin{aligned} K(U,W) &= -\frac{1}{2}V''(r)[U^rW^\theta - W^rU^\theta]^2 \\ &\quad -\frac{rV'(r)}{2}\left\{\frac{1}{V(r)}\|U^r\hat{W} - W^r\hat{U}\|^2_{\hat{g}} + V(r)\|U^\theta\hat{W} - W^\theta\hat{U}\|^2_{\hat{g}}\right\} \\ &\quad +r^2\hat{R}_{ABCD}U^AW^BU^CW^D - r^2V(r)(\|\hat{U}\|^2_{\hat{g}}\|\hat{W}\|^2_{\hat{g}} - \langle\hat{U},\hat{W}\rangle^2_{\hat{g}}) \\ &= -\frac{1}{2}V''(r)[U^rW^\theta - W^rU^\theta]^2 \\ &\quad -\frac{rV'(r)}{2}\left\{\frac{1}{V(r)}\|U^r\hat{W} - W^r\hat{U}\|^2_{\hat{g}} + V(r)\|U^\theta\hat{W} - W^\theta\hat{U}\|^2_{\hat{g}}\right\} \\ &\quad +r^2(\hat{K}(\hat{U},\hat{W}) - V(r))(\|\hat{U}\|^2_{\hat{g}}\|\hat{W}\|^2_{\hat{g}} - \langle\hat{U},\hat{W}\rangle^2_{\hat{g}}) . \end{aligned}$$

Set

$$(a,b,c) = (V^{-1/2}(r)U^r, V^{1/2}(r)U^\theta, r\hat{U}) ,$$

and

$$(x,y,z)=(V^{-1/2}(r)W^r,V^{1/2}(r)W^\theta,r\hat{W})\;,$$

so that $a^2+b^2+|c|^2=x^2+y^2+|z|^2=1$ and $ax+by+\langle c,z\rangle=0$, where the norm $|\cdot|$ and the scalar product $\langle\cdot,\cdot\rangle$ are taken with respect to $\hat{g}$. We can rewrite the sectional curvature as

$$\begin{aligned}K(U,W) &= -\frac{1}{2}V''(r)[ay-bx]^2\\ &\quad -\frac{r^{-1}V'(r)}{2}\left\{|az-xc|^2+|bz-yc|^2\right\}\\ &\quad +r^{-2}[\hat{K}(\hat{U},\hat{W})-V(r)](|c|^2|z|^2-\langle c,z\rangle^2)\;.\end{aligned}$$

Setting

$$k=\min(-\frac{1}{2}V''(r),-\frac{r^{-1}V'(r)}{2},r^{-2}[\hat{K}(\hat{U},\hat{W})-V(r)])\;,$$

we obtain

$$K(U,W)\geq k\left([ay-bx]^2+|az-xc|^2+|bz-yc|^2+|c|^2|z|^2-\langle c,z\rangle^2\right)=k\;.$$

Similarly,

$$K(U,V)\leq K:=\max(-\frac{1}{2}V''(r),-\frac{r^{-1}V'(r)}{2},r^{-2}[\hat{K}(\hat{U},\hat{W})-V(r)])\;.$$

Coming back to $V(r)=c+r^2-2mr^{-(n-3)}$, we further assume that $\hat{K}(\hat{U},\hat{W})=c$ and $n\geq 4$. One then finds

$$k=-1+r^{-(n-1)}\min\left\{(n-3)(n-2)m,-(n-3)m,2m\right\}\;,$$

so that if $m\geq 0$ then

$$k=-1-m(n-3)r^{-(n-1)}\;,\quad K=-1+m(n-3)(n-2)r^{-(n-1)}\;. \tag{D.2}$$

while for $m\leq 0$ one has

$$k=-1+m(n-3)(n-2)r^{-(n-1)}=-1-|m|(n-3)(n-2)r^{-(n-1)}\;, \tag{D.3}$$

$$K=-1+|m|(n-3)r^{-(n-1)}\;. \tag{D.4}$$

A) If $m\geq 0$, we have $k\geq -2(n-1)/n$ if and only if

$$mr_+^{-(n-1)}\leq\frac{n-2}{n(n-3)}\;, \tag{D.5}$$

where r_+ is the unique positive solution of

$$V(r_+)=0\quad\Longleftrightarrow\quad mr_+^{-(n-3)}=\frac{1}{2}\left(c+r_+^2\right)\;. \tag{D.6}$$

On the other hand, $K\leq 0$ will hold if and only if

$$mr_+^{-(n-1)}\leq\frac{1}{(n-3)(n-2)}\;. \tag{D.7}$$

Since the right-hand-side of (D.5) is larger than that of (D.7) for $n > 4$, with equality for $n = 4$, the former condition is less restrictive than the latter. For further use we note that

$$mr_+^{-(n-1)} = r_+^{-2} mr_+^{-(n-3)} = \frac{1}{2}\left(1 + \frac{c}{r_+^2}\right) . \tag{D.8}$$

a) For $c = 1$ the left-hand-side of (D.5) is strictly larger than $1/2$ for $m > 0$ by (D.8), while the right-hand-side is less than or equal to $1/2$ when $n \geq 4$, and our non-degeneracy criterion in terms of k does not apply. Similarly one finds that some sectional curvatures are always positive at $r = r_+$.

If we assume that the sectional curvatures of $\hat{g}$ are equal to $c = 1$, and that the manifold N^{n-2} carrying the metric $\hat{g}$ is compact, then $(N^{n-2}, \hat{g})$ is clearly of positive Yamabe type, and one can likewise attempt to use point (b) of Theorem 2.2 to prove non-degeneracy. Unfortunately, it turns out that the sectional curvature inequality there is always violated at r_+.

b) If $c = 0$ then $r_+ = (2m)^{1/(n-1)}$, giving $1/2 = 1/2$ for $n = 4$ in (D.5), without restrictions on m. However, for $n \geq 5$ the right-hand-side of (D.5) is always smaller than one half. Similarly the inequality of point (b) of Theorem 2.2 always fails.

c) If $c = -1$ then the map

$$[1, \infty) \ni r_+(m) \longleftrightarrow m(r_+) \in [0, \infty)$$

is a bijection, and for all $0 \leq m \leq m_+$ from (D.5) we obtain non-degeneracy, where $m_+ = \infty$ if $n = 4$. For $n \geq 5$ the value of m_+ can be found by first solving (D.5) in terms of r_+ using (D.8),

$$r_+(m_+) = \sqrt{\frac{n(n-3)}{n(n-3) - 2(n-2)}} = \sqrt{\frac{n(n-3)}{(n-1)(n-4)}} .$$

Equation (D.6) can then be used to calculate $m_+ = m_+(n)$:

$$m_+(n) = \begin{cases} \infty, & n = 4; \\ \frac{(n-2)}{(n-1)(n-4)} \left(\frac{n(n-3)}{(n-1)(n-4)}\right)^{\frac{n-3}{2}}, & n \geq 5. \end{cases} \tag{D.9}$$

B) If $m < 0$, we have $k \geq -2(n-1)/n$ if and only if

$$|m| r_+^{-(n-1)} \leq \frac{1}{n(n-3)} , \tag{D.10}$$

while $K \leq 0$ is equivalent to

$$|m| r_+^{-(n-1)} \leq \frac{1}{(n-3)} , \tag{D.11}$$

this last condition being less restrictive than (D.10).

The only case of interest is $c = -1$, as V has no zeros otherwise. The map

$$\left[r_{\min} := \sqrt{\frac{n-3}{n-1}}, 1\right] \ni r_+(m) \longleftrightarrow m(r_+) \in \left[\frac{1}{2}(r_{\min}^{n-1} - r_{\min}^{n-3}), 0\right]$$

is a bijection, and for all $m_- \le m < 0$ from (D.11) we obtain negative sectional curvatures, where

$$r_+(m_-) = r_{\min} = \sqrt{\frac{n-3}{n-1}}\,, \tag{D.12}$$

$$m_- = m_-(n) = -\frac{1}{n-1}\left(\frac{n-3}{n-1}\right)^{\frac{n-3}{2}}\,. \tag{D.13}$$

Recall that $r_{\min}$ given by (D.12) corresponds to the smallest value of $r_+(m)$ for which a regular solution exists. Equations (D.12)-(D.13) show that non-degeneracy holds in the whole range of negative masses compatible with a singularity-free metric. Summarising, we have proved:

PROPOSITION D.2 *Let $V(r) = -1 + r^2 - 2mr^{-(n-3)}$, suppose that $\hat{g}$ is a metric of constant sectional curvature equal to -1 on a compact manifold N^{n-2}, then for $n \ge 4$ and for*[7] *$m \in (m_-(n), m_+(n)]$, as given by* (D.13) *and* (D.9)*, the metric* (D.1) *is non degenerate. In dimension four all singularity-free such solutions are non-degenerate.*

References

[1] M.T. Anderson, *Einstein metrics with prescribed conformal infinity on* 4*-manifolds*, (2001), math.DG/0105243.

[2] ______, *Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on* 4*-manifolds*, Adv. in Math. **179** (2003), 205–249, math.DG/0104171.

[3] M.T. Anderson, P.T. Chruściel, and E. Delay, *Non-trivial, static, geodesically complete vacuum space-times with a negative cosmological constant*, JHEP **10** (2002), 063, 22 pp., gr-qc/0211006.

[4] L. Andersson and P.T. Chruściel, *On asymptotic behavior of solutions of the constraint equations in general relativity with "hyperboloidal boundary conditions"*, Dissert. Math. **355** (1996), 1–100.

[5] L. Andersson and M. Dahl, *Scalar curvature rigidity for asymptotically locally hyperbolic manifolds*, Annals of Global Anal. and Geom. **16** (1998), 1–27, dg-ga/9707017.

[6] T. Aubin, *Espaces de Sobolev sur les variétés Riemanniennes*, Bull. Sci. Math., II. Ser. **100** (1976), 149–173.

[7] A.L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 10, Springer Verlag, Berlin, New York, Heidelberg, 1987.

[7] The case $m = m_-(n)$ corresponds to $V'(r_{\min}) = 0$, which leads to a cylindrical end for the metric (D.1), and is therefore excluded by the requirement that the associated Riemannian manifold be conformally compact. The corresponding Lorentzian solution is a regular black hole with vanishing surface gravity.

[8] O. Biquard, *Métriques d'Einstein asymptotiquement symétriques (Asymptotically symmetric Einstein metrics)*, Astérisque 265, Paris: Société Mathématique de France, 109 pp., 2000.

[9] J. Bjoraker and Y. Hosotani, *Monopoles, dyons and black holes in the four-dimensional Einstein-Yang-Mills theory*, Phys. Rev. **D62** (2000), 043513, hep-th/0002098.

[10] G.E. Bredon, *Introduction to compact transformation groups.*, Pure and Applied Mathematics. Vol. 46. Academic Press, New York, 1972.

[11] U. Christ and J. Lohkamp, in preparation (2003).

[12] P.T. Chruściel, E. Delay, J.M. Lee, and D.N. Skinner, *Boundary regularity of conformally compact Einstein metrics*, (2003), math.DG/0401386.

[13] P.T. Chruściel and W. Simon, *Towards the classification of static vacuum spacetimes with negative cosmological constant*, Jour. Math. Phys. **42** (2001), 1779–1817, gr-qc/0004032.

[14] R. Coquereaux and A. Jadczyk, *Riemannian geometry, fiber bundles, Kaluza-Klein theories and all that*, World Sci. Lect. Notes Phys. **16** (1988), 1–345.

[15] E. Delay, *Essential spectrum of the Lichnerowicz laplacian on* 2*-tensors on asymptotically hyperbolic manifolds*, Jour. Geom. Phys. **in press** (2002).

[16] C. Fefferman and C.R. Graham, *Conformal invariants*, Élie Cartan et les mathématiques d'aujourd'hui, The mathematical heritage of Élie Cartan, Sémin. Lyon 1984, Astérisque, No.Hors Sér. 1985, 95-116.

[17] R. Fintushel, *Circle actions on simply connected 4-manifolds.*, Trans. Am. Math. Soc. **230** (1977), 147–171.

[18] ———, *Classification of circle actions on 4-manifolds.*, Trans. Am. Math. Soc. **242** (1978), 377–390.

[19] R. Geroch, *A method for generating solutions of Einstein's equations*, Jour. Math. Phys. **12** (1971), 918–924.

[20] G. W. Gibbons and S. W. Hawking, *Classification of gravitational instanton symmetries*, Commun. Math. Phys. **66** (1979), 291–310.

[21] C.R. Graham and J.M. Lee, *Einstein metrics with prescribed conformal infinity on the ball*, Adv. Math. **87** (1991), 186–225.

[22] S. de Haro, S.N. Solodukhin, and K. Skenderis, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence*, Commun. Math. Phys. **217** (2001), 595–622, hep-th/0002230.

[23] J.M. Lee, *Fredholm operators and Einstein metrics on conformally compact manifolds*, Memoirs AMS, in press, math.DG/0105046.

[24] ———, *The spectrum of an asymptotically hyperbolic Einstein manifold*, Comm. Anal. Geom. **3** (1995), no. 1-2, 253–271.

[25] M. Mars and W. Simon, *A proof of uniqueness of the Taub-bolt instanton*, Jour. Geom. Phys. **32** (1999), 211–226.

[26] R. Mazzeo, *The Hodge cohomology of a conformally compact metric.*, Jour. Diff. Geom. **28** (1988), 309–339.

[27] J. Qing, *On the uniqueness of AdS spacetime in higher dimensions*, Annales Henri Poincaré **5** (2004), 245–260, math.DG/0310281.

[28] W. Simon, *NUTs have no hair*, Class. Quantum Grav. **12** (1995), L125–L130, gr-qc/9508042.

[29] X. Wang, *On conformally compact Einstein manifolds*, Math. Res. Lett. **8** (2001), 671–688.

[30] E. Winstanley and O. Sarbach, *On the linear stability of solitons and hairy black holes with a negative cosmological constant: The even-parity sector*, Class. Quantum Grav. **19** (2002), 689–724, gr-qc/0111039.